Dual Labor Markets and the Equilibrium Distribution of Firms[*]

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Abstract

We study the effects of dual labor markets —namely, the co-existence of fixed-term and open-ended contracts— on the equilibrium distribution of firms, the allocation of workers across and within firms, and aggregate productivity. Using rich Spanish administrative data, we document that the use of fixed-term contracts is very heterogeneous across firms within narrowly defined sectors. Particularly, there is a strong relationship between the share of temporary workers and firms size, which is positive when looking at *within*-firm variation but negative when looking at the variation *between* firms. To capture these facts, we write a directed search model of multi-worker firms with fixed-term and open-ended contracts. Firms are ex-ante heterogeneous in their technology type, and ex-post heterogeneous in transitory productivity, level of employment, and the composition of employment in terms of both contract types and human capital accumulated on the job. In the calibrated model, the permanently more productive firms prefer a higher share of high human capital workers, so they employ a larger share of their workers with an open-ended contract. However, within a firm type, fixed-term contracts are increasingly useful for firms as they grow toward their optimal size. In counterfactual exercises, we find that limiting the duration of fixed-term contracts decreases the share of temporary employment, increases aggregate productivity, and reduces total employment for an overall decline in total output. The increase in productivity comes from a better selection of incumbent firms and an increase in the number of firms relative to total employment, which more than offsets an increased misallocation of workers across firms.

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1 Introduction

Many countries feature labor markets characterized by a two-tier system, with the co-existence of open-ended (OE) contracts with large termination costs and fixed-term (FT) contracts of short duration. The effects of labor market duality on individual and aggregate worker outcomes have been widely studied.[1] However, little is known about the firm-level determinants of differential contract use and the effect of this duality on the equilibrium distribution of firms, the allocation of workers across heterogeneous production units, or aggregate productivity.

This paper sets out to fill this gap, and ultimately aims to understand how changes in the menu of labor contracts may affect different firms differently and, through that, macroeconomic outcomes.[2] On the one hand, the existence of FT contracts in heavily regulated labor markets may ease the reallocation of workers across firms experiencing productivity shocks, leading to lower levels of misallocation. On the other hand, FT contracts may benefit more those firms whose technology relies less heavily on human capital and hence are less damaged by worker turnover, leading to worse firm selection. The main contribution of our paper is to provide a framework that allows us to quantitatively discipline these different channels.

We start by documenting new facts about the heterogeneous usage of FT contracts across firms. To do so, we exploit Spanish administrative firm-level data for the period 2004-2019, which reports the number of employees hired under each type of contract as well as various standard balance-sheet items and income account data. The case of Spain is of particular interest because it is a country where labor market duality is very stark, with a high incidence of FT contracts and a strong employment protection for OE contracts.[3] We document the following three facts. First, there is a large degree of heterogeneity in the usage of FT contracts across firms, with the distribution being very right-skewed. For instance, while the average FT share across firms is 18.1%, the firm at the median of the temporary share distribution employs just 2.7% of workers under FT contracts and the firm at the 90th percentile employs 60% of FT workers. Second, although the use of FT contracts varies greatly across narrowly-defined industries, provinces, and time, most of the variation occurs due to firm-specific factors within industry, province, and time period, of which time-invariant unobserved firm characteristics play an overwhelmingly relevant role. And third, exploiting the panel dimension of our data, we find that the *within-firm* variation uncovers a positive correlation

[1] By and large, workers in temporary positions have high turnover rates, earn lower wages, experience lower wage growth, receive less training on the job, and accumulate less human capital (see e.g. Blanchard and Landier (2002), Booth, Francesconi and Frank (2002), Güell and Petrongolo (2007), Autor and Houseman (2010), Bentolila, Dolado and Jimeno (2020), Bratti, Conti and Sulis (2021), and Garcia-Louzao, Hospido and Ruggieri (2023)).

[2] Though our approach is quantitative, there is flourishing empirical evidence showing that changes in various labor market policies have effects on the equilibrium distribution of workers and firms. For instance, Dustmann, Lindner, Schönberg, Umkehrer and vom Berge (2021), Luca and Luca (2019), and Chava, Oettl and Singh (2019) show that changes in minimum wages in Germany or the US reallocate workers towards larger and more productive firms, or increase the exit rates of smaller and less productive firms. Likewise, Carry (2022) shows that changes in the minimum workweek regulation in France led to distributional effects among both firms and workers.

[3] In 2019, the share of workers under an FT contract was 26.3% in Spain —the highest among OECD countries—, 21.8% in Poland, 20.3% in Portugal, and 20.3% in Netherlands (see https://data.oecd.org/emp/temporary-employment.htm).

between firm size and the share of temporary workers, while the *between-firm* variation shows the opposite relationship.

Equipped with these findings, we write a model of firm dynamics with search-and-matching frictions and a two-tier labor market structure to understand the firm-level determinants of the usage of FT contracts and the macroeconomic consequences of dual labor markets. In the model, a set of multi-worker firms operating a decreasing returns-to-scale technology can direct unemployed workers by posting (and committing to) dynamic long-term contracts. Firms differ in permanent technology type ex-ante and are subject to idiosyncratic productivity shocks through their life-cycle. A firm's technology type determines a permanent component of the firm's total productivity, as well as the relative productivity of workers of different firm-specific human capital levels. Contracts specify state-contingent trajectories for wages, layoff, and promotion rates, for the duration of a worker's tenure in the firm. Firms may simultaneously post two types of dynamic contracts, open-ended (OE) and fixed-term (FT), which differ in the rate at which worker-firm relationships come to an end. Hence, through the longer average duration of their contracts, workers employed under OE contracts can accumulate more firm-specific human capital. On top of this, we add firm entry and exit. Firms may exit the economy if they are hit by a destruction shock or if they lose their last remaining worker. Firms enter by drawing a type and hiring their first worker, and a free-entry condition endogenously pins down the aggregate measure of operating firms.

In equilibrium, unemployed workers remain ex-ante indifferent between applying for a job under either contract, as well as between the firms that offer them, because less ex-post attractive offers are posted in tighter markets. Similar to other directed search models, the menu of optimal contracts maximizes the joint surplus generated by the firm and its workers. We use this characterization to determine the rate at which new workers of each type are hired in equilibrium, which gives rise to a non-degenerate distribution of firms in the space of permanent firm productivity types, idiosyncratic productivity levels and the number of FT and OE workers within each firm.

We calibrate the model by Simulated Method of Moments to cross-sectional empirical patterns related to the distribution of firms and the share of temporary employment from our firm-level Spanish data, as well as to aggregate data on job flows into and out of unemployment by contract type. In order to accurately identify the parameters of the model, we group firms in the data into low and high permanent productivity groups (mirroring the two permanent types assumed in the model) using a k-means algorithm, and use an indirect inference approach to ensure that in the calibrated economy, as in the data, permanently large and productive firms have lower shares of temporary employment but, when looking at within-firm variation, firms employ an increasing share of their workers under a temporary contract as they grow larger. We further show that the parameters are well-identified (in a global sense) by their respective empirical targets when we follow this calibration strategy.

The calibration delivers two key results: (i) high-skilled workers are more productive in high-type firms (i.e. the relative productivity of high-skill workers vis-à-vis low-skill ones is higher in high-type firms), implying positive assortative matching between firms and workers; and (ii) for given market tightness, there are, on average, more matches per unit of time in the FT market than in the OE market. These two calibration results explain, through the lens of the model, the main empirical facts that we documented in the data. First, because of (i), high-type firms, which have a permanent productivity advantage, use OE contracts more extensively to retain a larger share of their workers and enhance their chances of accumulating human capital on the job. In the cross-section of firms, this generates the inverse *between-firm* relationship between firm size and temporary share that we see in the data. Second, because of (ii), as firms grow they accumulate more FT workers, conditional on a given productivity type, which generates the positive *within-firm* relationship between firm size and temporary share that we see in the data. This is because, at a given productivity, firms face a trade-off between (a) the lower costs of attracting workers to FT contracts (due to the higher job-filling rates, for given tightness, in the FT market), and (b) the higher worker turnover of FT positions. This leads to different incentives to hire FT workers as the firm expands or contracts in the size space. On the one hand, the larger the firm is, the lower its opportunity cost of leaving a vacancy unfilled because, due to the presence of decreasing returns to scale, the marginal product of labor declines in firm size. This force pushes toward a higher FT share. On the other hand, however, a larger firm with a higher temporary share will face a more frequent attrition of workers. The positive *within-firm* relationship between firm size and temporary share is driven by the fact that, as firms reach their optimal size, the hiring cost effect dominates the turnover effect.

Next, we seek to quantify the degree of employment misallocation in the calibrated economy. We show that the calibrated economy exhibits a significant degree of misallocation of workers across firms, with aggregate productivity being 6.8% lower than in the allocation of a social planner that is not constrained by search frictions and faces the same distribution of firms as in competitive equilibrium. About half of this loss is due to misallocation between firms of the same productivity level, coming from the fact that, in the competitive equilibrium, there is dispersion in the level and composition of employment across equally productive firms. The other half of the losses are due to misallocation between firms of different productivity levels, with the most productive firms not receiving enough employment and with the degree positive assortative matching between worker skills and firm productivity types being too low relative to the planner's allocation.

Finally, we use the calibrated model for counterfactual analysis to understand the effects of a dual labor market structure on the distribution of firms across different sizes and productivities, and of workers within and across firms. Particularly, in line with some recent labor market reforms that attempt to limit the use of FT contracts to induce firms into using more OE contracts, we change the duration of FT contracts and compare across different stationary solutions. We find that reducing the duration of FT contracts does indeed reduce the share of temporary employment. Moreover, the policy leads to an increase in the aggregate productivity of the economy. However, this comes at the expense of a decrease in the employment rate that is of a larger magnitude than the increase in productivity, leading to an overall output decline.

The reduction in the share of temporary contracts is intuitive: shorter FT duration increases worker turnover associated to these contracts, which makes them less attractive to firms. Firms react by hiring proportionately more from the OE market and increasing the promotion rate of their incumbent FT workers. The effect on the employment-to-unemployment (EU) flows is in principle ambiguous. The EU flow increases (mechanically) among FT workers, but the decrease in the temporary share decreases the aggregate EU flow due to a composition effect. In our calibration the former effect dominates and the aggregate EU flow increases, which pushes up the unemployment rate. Moreover, as firms are now composed of lower-turnover workers, unemployment-to-employment (UE) rates decrease, increasing the average duration of unemployment spells and contributing to a further increase in overall unemployment.

Aggregate productivity increases when FT contracts become of shorter duration. To understand this effect, we decompose aggregate productivity into three components: a *firm size* term, a *firm selection* term, and a *reallocation* term. First, as FT contract duration decreases, the value of firm entry declines, which lowers the mass of operating firms. However, because total employment falls even more, the mass of firms per worker increases (firms become smaller on average) which, under decreasing returns to scale, leads to aggregate productivity gains (by 2.1% for a reduction in FT contract duration from 5 months to 1 month). Second, the share of high-type firms increases in equilibrium, because low-type firms, who are more reliant on low-skilled (and thus, predominantly FT) workers, face a higher risk of exiting in response to the increase in FT worker turnover rates that is induced by the policy. As high-type firms are more productive overall, this firm selection channel contributes to increasing aggregate productivity. And third, the distribution of workers across firms of different productivities changes in a number of ways in response to the policy change. Within those, we distinguish between (i) within-firm reallocation of human capital, and (ii) between-firm reallocation of total employment. On the one hand, for given total firm size and productivity, the allocation of workers within the firm improves (the share of high-skilled workers increases on average across all productivity groups), contributing positively to aggregate productivity. On the other hand, for a given within-firm composition of skills, employment is inefficiently reallocated away from more productive firms relative to the planner's solution. That is, when FT contracts are less useful and firms rely more on OE contracts, it is harder for firms to reallocate workers as they experience positive and negative productivity shocks. Overall, the latter effect dominates and the reallocation effect worsens aggregate productivity.

Related Literature There is a large literature studying the effects of the duality of employment contracts on the labor market outcomes of workers, like the average unemployment rate, the volatility of employment, or the dynamics of labor market flows.[4] Blanchard and Landier (2002), Cahuc and Postel-Vinay (2002), Bentolila, Cahuc, Dolado and Le Barbanchon (2012), and Sala, Silva and Toledo (2012) study the effect of dual labor markets in models with search and matching frictions *à la* Mortensen and Pissarides (1994). In these models, search is random and firms do not choose which type of contracts to offer. If they did, they would hire all new workers in FT contracts as in Costain, Jimeno and Thomas (2010) because from the firm side the flexibility of

[4] For a worker-side analysis in the case of Spain, see Dolado, García-Serrano and Jimeno (2002).

FT contracts dominates OE contracts.[5] Hence, these models are not designed to understand the differential choices of FT vs OE contracts across firms. Furthermore, because firms can only hire one worker, they cannot be used to link contract choices to firm dynamics. Our paper contributes to this literature on both of these fronts.

Several papers provide arguments for the co-existence of FT and OE contracts. In a similar framework as the papers above, Cahuc, Charlot and Malherbet (2016) allow for firms to be heterogeneous in their expected job duration and to choose the type of contract associated to their vacancies. This serves to show that firms prefer to use FT contracts for jobs of short expected duration (in order to save on firing costs) and OE contracts for jobs of long expected duration (to save on vacancy posting costs). In the context of directed search models, Berton and Garibaldi (2012) argue that an advantage of OE over FT contracts for firms is that the vacancy-filling rate will be higher in equilibrium when posting OE contracts as more job-seekers will self-select into the OE market, which offers them higher-value jobs ex-post. Our model features a similar equilibrium logic. However, by properly parameterizing the matching function of the OE and FT markets, we can obtain vacancy-filling rates that are larger in the FT than OE markets, a feature that is needed to match the large gap in labor market flows between contract types in the data. Other explanations for the coexistence of OE and FT contracts are that duality diminishes on-the-job search and hence allows firms to retain high-quality workers (as in Cao, Shao and Silos (2013)), and that it can be used by firms to overcome their financial contraints (as in Caggese and Cuñat (2008)).

As we analyze dual labor markets from the firm's side, we also relate to a macro literature studying the equilibrium dynamics of multi-worker firms in the context of frictional labor markets. We use a directed search framework with dynamic long-term contracts in the spirit of Kaas and Kircher (2015) and Schaal (2017). We adapt this framework to a continuous-time setting with slow-moving state transitions similar to Roldan-Blanco and Gilbukh (2021), and extend it to incorporate segmented labor markets, different ex-ante firm types, and double-sided ex-post heterogeneity. An alternative approach in the literature has been to assume random search and study firm dynamics in the context of decreasing returns and Nash bargaining (as in Elsby and Michaels (2013) or Acemoglu and Hawkins (2014)), or in setting with on-the-job search and a variety of wage-setting protocols (e.g. Moscarini and Postel-Vinay (2013), Coles and Mortensen (2016), Bilal, Engbom, Mongey and Violante (2019), Gouin-Bonenfant (2020), Audoly (2020) and Elsby and Gottfries (2022)). These models have been shown to provide an outstanding quantitative fit of the labor market flows and the hiring and firing decisions of firms in non-dual markets. Our contribution to this literature is to provide a quantitative model for the firm and aggregate labor market dynamics of tiered markets.

On the empirical side, several papers have studied the effects of employment protection legislation in the context of dual labor markets. For example, Daruich, Di Addario and Saggio (2020) show that relaxing constraints on FT contracts relative to OE contracts in Italy failed to increase overall employment, and that it increased firms' profits mostly at the expense of young workers. Using data

[5] Still, the endogenous firing decision in these models allows to make separations contingent on contract type and hence the aggregate share of FT contracts is determined in equilibrium.

from Portugal, Cahuc, Carry, Malherbet and Martins (2022) show that a policy intended to restrict the use of fixed-term contracts by new establishments of large firms did not increase the number of permanent contracts and ended up decreasing employment in large firms. We complement these studies by showing that, within the context of our calibrated model, changing the duration of fixed-term contracts yields significant effects on aggregate productivity and employment that mask various selection and reallocation effects.

Finally, even though dual labor markets are typically associated to European economies due to certain institutional arrangements (such as the availability of different contract types, unions, size-dependent policies, or other employment protection legislation), recent papers have documented a de facto duality in the U.S. labor market as well (e.g. Gregory, Menzio and Wiczer (2022) and Ahn, Hobijn and Şahin (2023)). Given this, we view our theory, and particularly our productivity decomposition and counterfactual results, as potentially useful for the U.S. context as well.

Outline The rest of the paper is organized as follows. Section 2 describes the data and our main empirical findings. Section 3 outlines the model and characterizes its equilibrium conditions. Section 4 discusses the estimation and presents a few counterfactual experiments. Section 5 describes our main policy experiment. Section 6 offers concluding remarks. Proofs and additional results, tables, and figures can be found in the Appendix.

2 Empirical Findings

Data We use annual data for Spain from the *Central de Balances Integrada* (CBI) dataset, a comprehensive and unbalanced panel of confidential firm-level balance-sheet data compiled and processed by the *Central de Balances*, a department within Banco de España.[6] This dataset covers the quasi-universe of Spanish firms, including large and small firms as well as privately held and publicly traded firms. Among many other items from the balance sheet of firms, the data provide information on total employment and the type of employment contract. The data has excellent geographical coverage and is representative of all non-financial sectors of economic activity (excluding the primary and public sectors). Industries are defined up to 4-digit codes in the NACE Rev. 2 classification.

We use data for the period 2004-2019.[7] We restrict our sample to firms observed for at least 5 years and whose average employment over the period is at least one worker. After some cleaning, we keep data for 7,153,669 firm-year observations, corresponding to 705,879 different firms. The aggregation of our firm-level data for the temporary share is quite similar to the numbers one can

[6]All Spanish firms are obliged by law to report their end-of-year accounting results to the commercial registry of the region where the company is registered. The *Central de Balances* obtains these data from the commercial registries directly, and homogenizes them into a unique dataset. In order to further expand the coverage of the data, Banco de España merged CBI with the SABI database (Iberian Balance-Sheet Analysis System), owned by Informa-Bureau van Dijk, which is used as Spain's input to the widely-used Amadeus and Orbis datasets. Detailed information about the CBI-SABI data set and its representativeness is provided by Almunia, López-Rodríguez and Moral-Benito (2018).

[7]We start in 2004 because earlier years of the sample do not track well the aggregate share of temporary workers from the labor force survey.

Table 1: Temporary share, descriptive statistics

Firm size (employment)	**Share of firms** (%)	Mean	p10	p25	p50	p75	p90	p95
		Distribution of firm-level share of temporary employment						
Total	**100**	**0.181**	**0.000**	**0.000**	**0.027**	**0.294**	**0.591**	**0.800**
1-10	77.65	0.164	0.000	0.000	0.000	0.250	0.541	0.776
11-50	19.04	0.250	0.000	0.031	0.163	0.391	0.677	0.825
51-100	1.78	0.255	0.000	0.034	0.160	0.393	0.701	0.861
101-200	0.99	0.237	0.000	0.029	0.147	0.361	0.645	0.833
201-500	0.30	0.222	0.000	0.026	0.137	0.329	0.589	0.796
501-1,000	0.14	0.229	0.000	0.033	0.142	0.333	0.616	0.841
1,001-5,000	0.10	0.258	0.000	0.048	0.165	0.374	0.746	0.947
5,000+	0.02	0.279	0.000	0.063	0.191	0.393	0.709	0.964

obtain from more conventional labor force survey data.[8] Our companion paper, Auciello-Estévez, Pijoan-Mas, Roldan-Blanco and Tagliati (2023), describes the data in more detail and expands on some of the empirical results presented below.

Cross-sectional distribution We start by looking at the distribution of the temporary share of workers across firms. In our sample, the average share of temporary workers across all years is 18.1%, while the median is 2.7% (see first row in Table 1). This gap reflects a highly right-skewed distribution. Part of this skewness is due to a large incidence of very small firms (1 or 2 workers) with no temporary workers. However, when we look at the temporary share within firm size, the highly right-skewed distribution is still apparent (see rows 2 to 8 in Table 1). For instance, within the subset of firms between 11 and 50 workers, the average share of temporary workers is 25.0%, and the share of temporary workers at the 25th, 50th and 90th percentiles is 3.1%, 16.3%, and 67.7%, respectively. Thus, a relatively small fraction of firms make very intensive use of fixed-term contracts compared to the median firm.[9] Finally, in Table 1 we also observe that the use of temporary workers tends to increase with firm size —something that we will discuss in more detail later— and that most of Spanish firms (more than 95%) have 50 workers or less.

Aggregate determinants Next, in order to understand the heterogeneity in the use of temporary contracts across firms, we propose a regression of the type:

$$TempSh_{ft} = \underbrace{(\alpha_i + \alpha_p + \alpha_t)}_{\text{Aggregate fixed effects}} + \alpha_f + \boldsymbol{X}_{ft}\boldsymbol{\beta} + \epsilon_{ft} \tag{1}$$

[8] In particular, in Appendix Figure D.1 we plot the share of workers under temporary contract for the years 2004-2019 from the Encuesta de Población Activa (EPA) and the employment-weighted average of the temporary share across firms in our data, which corresponds to the temporary share across workers. The time variation in the EPA and in the firm-level data is very similar. In terms of levels, there is only a difference of 3 to 5 percentage points between the two.

[9] The distribution of temporary share is also highly right-skewed across different firm age and productivity groups. The interested reader is referred to the complete analysis in Auciello-Estévez *et al.* (2023).

where $TempSh_{ft}$ is the share of temporary workers of firm f at time t; α_i, α_p, and α_t, are 4-digit industry, province, and year fixed effects; α_f are firm fixed effects; and $\boldsymbol{X}_{ft}$ is a vector of firm-level covariates.[10]

We find that the aggregate fixed effects (α_i, α_p, α_t) are important. The temporary share differs widely across sectors (ranging from 7.7% in "*Real estate activities*" to 43.1% in "*Employment activities*"), across provinces (ranging between 11.5% in Barcelona to 38.8% in Huelva), and over time (the temporary share is strongly procyclical, ranging from 23.9% in 2006 to 15.8% in 2012). However, the R^2 of regression (1) with only the aggregate fixed effects $(\alpha_i, \alpha_p, \alpha_t)$ is 16%. That is, 84% of the variation in the usage of fixed-term contracts remains within industry, province, and time period. In particular, firm fixed effects explain nearly half of the overall variation: the R^2 of regression (1) with aggregate fixed effects increases from 16% to 62% with the inclusion of α_f into the specification.

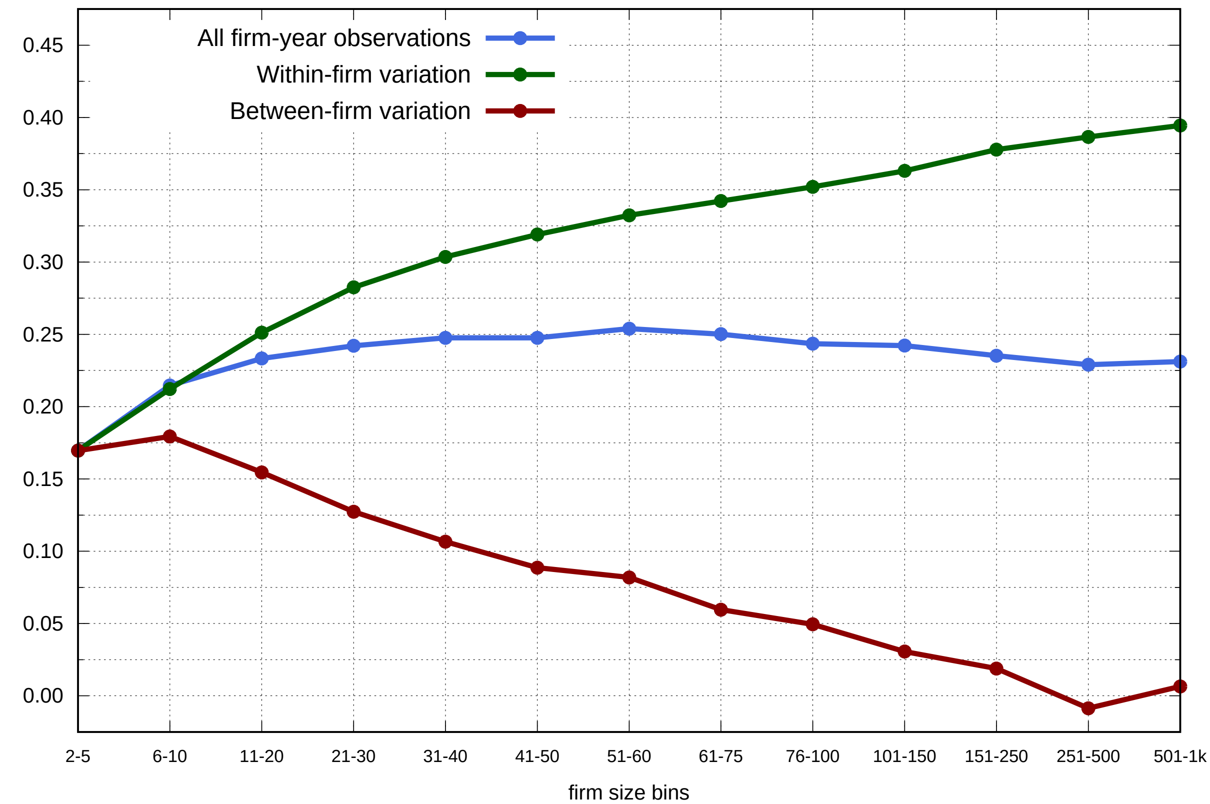
Temporary contracts and firm size Finally, we run regression (1) with size-bin dummies (2-5 employees, 6-10, 11-20, 21-30, ...) in the vector $\boldsymbol{X}_{ft}$ together with the aggregate and firm-level fixed effects. We find that there is a complex relationship between the use of temporary contracts and firm size: when looking at the within-firm variation ($\boldsymbol{\beta}$), the share of temporary contracts increases with firm size; however, when looking at the between-firm variation (α_f), the relationship between the share of temporary contracts and firm size is reversed.

The within-firm variation suggests that firms make use of FT contracts to grow or decline in size, that is, at the firm level FT contracts are useful to help employment track productivity or demand changes. The between-firm variation suggests that there are important technology differences across firms, whereby firms that are (permanently) larger prefer a lower share of temporary contracts, while firms that are (permanently) smaller prefer a higher share of temporary contracts. These patterns can be appreciated in Figure 1. The green line represents the estimated $\boldsymbol{\beta}$ coefficients in the regression with both aggregate and firm fixed effects included, and it captures an increasing within-firm variation between the temporary share and firm size. The red line plots the estimated α_f against the (time-series) average of employment of each firm, and it captures a declining between-firm variation between the temporary share and firm size. Both lines are scaled to deliver the same value for the size bin of 2-4 workers. For comparison, the blue line represents the estimated $\boldsymbol{\beta}$ coefficients in a regression that does not include firm fixed effect, and hence it does not distinguish between within- and between-firm variation. This relationship is a mix of the previous two, and shows that the share of temporary workers increases with firm size up to 60 workers, and it declines mildly afterwards.

Taking stock Our empirical analysis shows that most variation in the temporary share across firms is explained by firm-specific factors, and not by industry, province, or year effects. We have seen

[10] We index each data point by ft instead of $ipft$ since there is no variation in industry i and (almost) no variation in province p at the firm level.

Figure 1: Temporary share, by firm size: within- and between-firm variation.



Notes: The green line reports the coefficients of the size dummies of a regression of temporary share that controls for aggregate and firm-level fixed effects. The red line reports the firm fixed effects of the same regression against dummies of average firm size. The blue line reports the size dummies of a regression of temporary share that controls for aggregate but not firm-level fixed effects.

that firms use temporary contracts when they grow or decline (within-firm, the temporary share increases with employment), but that firms that are larger make a lower use of temporary contracts (between-firm, the temporary share declines with employment). With these results in mind, in the next section we write a firm-dynamics model with a dual labor market structure which we calibrate to replicate these and other features of the data in order to study the implications of labor market duality for the aggregate economy.

3 Model

3.1 Environment

Demographics Time is continuous, infinite, and indexed by $t \in \mathbb{R}_+$. We consider a stationary environment with no aggregate shocks. The economy is populated by a mass of workers with fixed unit measure and an endogenous measure $F > 0$ of firms. The measure of firms is an equilibrium object determined by a free-entry condition. Both firms and workers are risk-neutral and infinitely-lived, and share a common time discount rate, $\rho > 0$.

Workers and firms interact in frictional labor markets. Workers are ex-ante identical. Ex-post, they may be unemployed and receiving a flow unemployment benefit $b > 0$, or employed by a firm under different types of contracts which provide them with a wage. Ex-post, employed workers may also differ in their level of skills. These different sources of ex-post heterogeneity are described in detail below.

Technology At birth, a firm draws its type, denoted by $\varphi \in \Phi \equiv \{\varphi_1, \varphi_2\}$, from a probability mass function π_φ, with $\pi_\varphi(\varphi) \geq 0$, $\forall \varphi \in \Phi$, and $\sum_\varphi \pi_\varphi(\varphi) = 1$. This type is kept fixed through the firm's life. As detailed below, a firm's type determines (i) a permanent component in firm-level productivity, and (ii) the level of worker-specific productivity by skill type.

There are two possible worker skill levels in the firm, high (H) and low (L). Let $n_j = 0, 1, 2, \ldots$ denote the number of workers with skill $j \in \mathcal{J} \equiv \{H, L\}$ in a firm. The production technology is a constant-elasticity-of-substitution (CES) composite of skill types, so that:

$$Y(n_H, n_L, z, \varphi) = e^{z+\zeta(\varphi)} \Big(\omega(\varphi) n_H^\alpha + \big(1 - \omega(\varphi)\big) n_L^\alpha \Big)^{\frac{\nu}{\alpha}} \tag{2}$$

where $\alpha < 1$ and $\nu \in (0, 1]$ are parameters that are common across firms and, for each $\varphi \in \Phi$, we have $\omega(\varphi) \in (0, 1)$ and $\zeta(\varphi) > 0$.

Equation (2) states that employed workers of different skills are imperfect substitutes within the firm, with the elasticity of substitution between worker types equal to $\frac{1}{1-\alpha} > 0$. The ν parameter measures the degree of diminishing returns to scale in technology, and skill types are complements in production if $\alpha < \nu$.[11] The parameter $\omega(\varphi)$ measures the relative productivity of high-skilled workers for a firm of permanent type φ, whereas $\zeta(\varphi)$ denotes this firm type's *permanent* productivity level. The random variable $z \in \mathbb{Z} \equiv \{z_1 < \cdots < z_k\}$, in turn, stands for the *idiosyncratic and transitory* productivity level of the firm. We assume z follows a continuous-time Markov chain, and let $\lambda(z'|z)$ denote the intensity rate of a z-to-z' transition.[12] In Section 4.4, we will discuss how we quantitatively identify, using our firm-level data, the relationship between relative productivity $\omega(\cdot)$ and permanent productivity $\zeta(\cdot)$ for each permanent firm type φ. To solve for the equilibrium of the model, however, we impose no correlation structure on these variables.

Contracts, Skills, and Worker Flows Firms hire workers by posting contracts in the labor market. All starting positions are in low-skilled jobs and, as in Ljungqvist and Sargent (2007), workers can access high-skilled positions through skill upgrades that arrive on the job, as we describe below.

Firms can offer two types of contract, called fixed-term (FT) and open-ended (OE), which we index by $i \in \mathcal{I} \equiv \{FT, OE\}$. A worker is uniquely identified by $(i, j) \in \mathcal{I} \times \mathcal{J}$, i.e. by the contract i under which she is employed and the skill j under which she performs the job at a given point in

[11] In this case, the production function is supermodular, i.e. $\partial^2 Y / \partial n_H \partial n_L > 0$, $\forall z, \varphi$.

[12] The transition rates satisfy standard properties: for all $z \in \mathbb{Z}$, we have (i) $\lambda(z|z) \leq 0$; (ii) $\lambda(z'|z) \geq 0$, $\forall z' \neq z$, (iii) $\sum_{z'} \lambda(z'|z) = 0$; and (iv) $\sum_{z'} \lambda(z'|z) < +\infty$.

time. Either contract can be offered for any employee who first starts working for the firm. However, only OE workers can upgrade to high-skilled jobs. In particular, a low-skill OE worker transitions to a high-skill job with intensity $\tau > 0$, with the opposite transition (skill obsolescence) being impossible.[13] For simplicity, we treat τ as a parameter. Therefore, since all starting positions are low-skilled, high-skill workers can only be employed under an OE contract. Denoting by $n_{ij} = 0, 1, 2, \ldots$ the number of workers of type (i, j), then the number of low-skilled workers is $n_L \equiv n_{FT} + n_{OEL}$ and the number of high-skilled workers is $n_H \equiv n_{OEH}$.

Our assumption that only OE workers have access to the human capital accumulation technology has two possible interpretations. The first one is worker training by the firm, which is much more likely to be offered to workers under OE than FT contracts due to the higher turnover of the latter type of worker.[14] The second interpretation is learning-by-doing human capital accumulation, which workers under FT contracts do not have time to accumulate due to their short tenure at the firm.[15] Under either interpretation, there is an important motive to offer OE contracts to workers: it is the only way to ensure that a fraction of workers know the functioning of the firm and can be entrusted with tasks requiring a higher skill.

Firms may choose to promote their FT (low-skill) workers into an OE contract, with the opposite transition being prohibited by law. In particular, firms choose a promotion rate p for each one of their FT workers (if any), which carries a convex promotion cost:

$$C^P(p, n_{FT}) = \xi n_{FT} p^{\vartheta} \tag{3}$$

units of firm output, with $\xi > 0$ and $\vartheta > 1$. When an FT worker is promoted into an OE contract, her job description does not change (i.e. she continues to be a low-type worker). However, all low-skilled workers with an OE contract are exposed to skill upgrades, so $C^P(p, n_{FT})$ can be interpreted as training costs.

Firms may lose workers for three different reasons: (i) because of an exogenous firm exit shock, with intensity $s^F \geq 0$, dissolving the firm entirely and sending all of its workers into unemployment;

[13] As all starting jobs are low-skilled, we interpret human capital as being job-specific. This assumption allows us to simplify the worker side of the model considerably, as all unemployed individuals are identical (firm-specific human capital is lost upon displacement). Our assumption is supported by an empirical literature highlighting the importance of firm-specific human capital for worker wage growth (see e.g. Topel (1991), Dustmann and Meghir (2005), or Buchinsky, Fougère, Kramarz and Tchernis (2010)). Alternatively, we could have assumed that skills are attached to the workers themselves. In this case, to obtain a non-degenerate skill distribution, we would need to assume skill depreciation either upon separation or during unemployment (as in models of turbulent labor markets, e.g. Baley, Ljungqvist and Sargent (2022) and Baley, Figueiredo, Mantovani and Sepahsalari (2023)). While this is technically possible in our current framework, it would substantially complicate the quantitative implementation of the model without much gain in economic insight.

[14] Several papers show that having a temporary contract diminishes the probability of receiving on-the-job training (e.g. Alba-Ramirez (1994), Dolado, Felgueroso and Jimeno (2000), and Bratti *et al.* (2021)). For example, Cabrales, Dolado and Mora (2017) find that temporary workers are, on average, 16.4 percentage points less likely to receive on-the-job training. Additional evidence of this mechanism is that wage returns to experience are much larger for experience years accumulated under OE than FT contracts (see e.g. Garcia-Louzao *et al.* (2023)).

[15] As we discuss below, our calibrated average worker separation rates are 60% and 4.3% per quarter for FT and OE workers, respectively, which gives the former type of worker very little time (about 5 months) to acquire firm-specific skills. A similar mechanism is explored by Lafuente (2023) to explain state-dependence in the duration of unemployment spells.

(ii) because the contract expires, at rate $s_i^W \geq 0$ for each contract type $i \in \mathcal{I}$; or (iii) because the firm endogenously decides to fire workers. In particular, the firm must choose a per-worker firing rate $\delta_{ij} \geq 0$ for each worker $(i,j) \in \mathcal{I} \times \mathcal{J}$. A firing rate δ_{ij} carries a layoff convex cost equal to:

$$C^F(\delta_{ij}, n_{ij}) = \chi n_{ij} \delta_{ij}^{\psi} \tag{4}$$

units of firm output, with $\chi > 0$ and $\psi > 1$.[16] This cost is meant to capture expenses associated to laying off workers such as administrative expenses and legal costs.[17]

In the event that a firm loses all of its workers and becomes inactive, it must exit the market and become a so-called potential entrant. Potential entrants must incur a cost $\kappa > 0$ in order to attract their first worker. Upon successful entry, they draw both their permanent type $\varphi \in \Phi$ as well as an initial idiosyncratic productivity $z_e \in \mathbb{Z}$ from some probability mass function $\pi_z(z_e)$, with $\pi_z(z_e) \geq 0$ for all $z_e \in \mathbb{Z}$ and $\sum_{z_e \in \mathbb{Z}} \pi_z(z_e) = 1$.

Labor Markets Search is directed. Every period, firms publicly and simultaneously announce long-term FT and OE contracts in order to attract new workers. To simplify the vacancy-posting problem, we assume that the first job posting in each of the two markets is free and that any subsequent posting is prohibitively costly, so that firms can only post one vacancy per contract per instant of time.[18] Unemployed workers can perfectly observe the terms of all posted contracts (there are no informational asymmetries).

Denote the employment vector of a firm by $\vec{n} \equiv (n_{OEH}, n_{OEL}, n_{FT}) \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all possible triplets of integers excluding the zero vector, $\{(0,0,0)\}$. Let $(\vec{n}_t^{t+s}, z_t^{t+s})$ be the full history of possible firm states between t and $t+s$. At any date t and for all $s > 0$, a contract of type $i \in \mathcal{I}$ for job type $j \in \mathcal{J}$ offered by a firm of permanent type $\varphi \in \Phi$ is a set of complete state-dependent sequences of wages $w_{ij}(\vec{n}_t^{t+s}, z_t^{t+s}; \varphi)$, firing rates $\delta_{ij}(\vec{n}_t^{t+s}, z_t^{t+s}; \varphi)$ and, only for workers employed under FT contracts, intensities $p(\vec{n}_t^{t+s}, z_t^{t+s}; \varphi)$ of promotion into a OE contract, conditional on no worker separation and firm survival.

We assume the following commitment structure. On the worker's side, there is no commitment: workers may forfeit their contract and quit the firm, but they must go back to costly search if they seek to regain employment. On the firm's side, there is full commitment to both the contract type as well as to the contractual terms of this contract type, which cannot be revised or renegotiated for the duration of the match. Therefore, contracts must always comply with the firm's initial promises. Moreover, we assume that the firm cannot discriminate between workers with the same contract type and job description, i.e. all n_{ij} workers of type (i,j) obtain the same contract (though, of course,

[16] Notice that, mirroring Spanish law, we do not allow firing cost parameters to depend explicitly on contract type, implying that differences in firing costs between the two across firms will be an endogenous outcome of the model.

[17] The cost, however, does not include pure transfers between the employer and the worker (severance payments). A severance payment in this model would be a pure lump-sum transfer which would have no affect on the equilibrium allocation, as the optimal contract maximizes the joint surplus of the firm and all of its workers.

[18] Extending the model to include multiple contract-specific vacancy postings per instant of time is analytically possible, but computationally expensive.

their employment histories may differ).[19]

Given these assumptions, a labor market segment may be summarized by the long-term value that the worker can expect to obtain from it, denoted by W. Each firm can simultaneously post, and each worker can simultaneously search for, at most one offer. Let $f(W)$ be the measure of firms posting value W, $u(W)$ the number of unemployed workers applying for it, and denote this market's tightness by $\theta(W) = f(W)/u(W)$. The frequency of meetings is determined by a constant-returns-to-scale matching function $\mathcal{M}_i(f, u)$, whose parameters are constant across firms and possibly (though not necessarily) specific to each labor market $i = OE, FT$.[20] Given a market tightness $\theta \equiv f/u$, unemployed workers match with a firm on a contract of type i at Poisson rate $\mu_i(\theta) \equiv \mathcal{M}_i(\theta, 1)$, while firms find a worker for that same contract with Poisson intensity $\eta_i(\theta) \equiv \mathcal{M}_i(1, \theta^{-1})$, so that $\mu_i(\theta) = \theta\eta_i(\theta)$. We assume μ_i is increasing and concave, η_i is decreasing and convex, and the usual Inada conditions apply: $\mu_i(0) = \lim_{\theta\to+\infty} \eta_i(\theta) = 0$, and $\lim_{\theta\to+\infty} \mu_i(\theta) = \lim_{\theta\to 0} \eta_i(\theta) = +\infty$.

Recursive Formulation Because contracts are seemingly large and complex objects, we focus on a symmetric Markov Perfect Equilibrium, where the problem can conveniently be made recursive. Within this class of equilibria, contracts are functions of the firm's state, $(\vec{n}, z, \varphi) \in \mathbb{N} \times \mathbb{Z} \times \Phi$, and the vector of outstanding promises, $\vec{W} = (W_{OEH}, W_{OEL}, W_{FT}) \in \mathbb{R}^3_+$, that the firm promised to deliver to the workers that is currently employing. Precisely, the recursive contracts $\left(c_{ij}\right)_{(i,j)\in\mathcal{I}\times\mathcal{J}}$ are defined by the following array:[21]

$$c_{ij} = \left\{w_{ij}, \delta_{ij}, p, W'_{ij}(\vec{n}', z')\right\}$$

For each worker type $(i, j) \in \mathcal{I} \times \mathcal{J}$, each contracts includes a wage w_{ij}, a per-worker layoff rate δ_{ij}, a promotion rate p (for FT contracts only), and a continuation promise $W'_{ij}(\vec{n}', z')$ for each new possible set of states $(\vec{n}', z')$ of the firm. The possible new states, conditional on firm survival, are:

$$(\vec{n}', z') \in \left\{ \begin{array}{l} (n_{OEH},\ n_{OEL}+1,\ n_{FT},\ z),\ (n_{OEH},\ n_{OEL},\ n_{FT}+1,\ z), \\ (n_{OEH}+1,\ n_{OEL}-1,\ n_{FT},\ z) \\ (n_{OEH}-1,\ n_{OEL},\ n_{FT},\ z),\ (n_{OEH},\ n_{OEL}-1,\ n_{FT},\ z),\ (n_{OEH},\ n_{OEL},\ n_{FT}-1,\ z), \\ (n_{OEH},\ n_{OEL}+1,\ n_{FT}-1,\ z), \\ \left\{(n_{OEH},\ n_{OEL},\ n_{FT},\ z'),: z' \in \mathbb{Z}\right\} \end{array} \right\}$$

The first line says that the firm may hire a new low-skilled worker from unemployment using

[19]As it is standard in this type of models (e.g. Schaal (2017)), assuming worker discrimination would lead to wage indeterminacy, because the distribution of utilities across workers within a type is irrelevant for determining the optimal contract. Assuming no discrimination within worker types is only one of many ways to pin down wages.

[20]It should be noted that having contract-specific matching function parameters is not an essential assumption of our theory, in the sense that we can obtain differences in job-filling rates across contracts endogenously, even when both markets share a common matching function. However, introducing contract-specific matching technologies will help us quantitatively match the large gap in unemployment-to-employment worker flow rates across contracts that we see in the data. For a discussion, see Section 4.4.

[21]To alleviate notation, we do not index recursive contracts by firm type φ, as this is fixed through the firm's life.

either contract type. The second line states that a low-skill worker may become high-skill as long as she was employed as an OE worker (and therefore exposed to the τ shock). The third line shows that firms may lose a worker of any type, because the worker is fired or the contract expires. The fourth line states that a FT worker may be promoted to an OE contract, though in that case her job type remains low-skill. Finally, the firm may get a shock to her idiosyncratic productivity (fifth line). Since contracts are complete, prospective worker and firm agree on wages, firing rates, a promotion rate and continuation payoffs under each and every one of these possible contingencies.

We are now ready to characterize the equilibrium of this economy. To proceed, we first pose the value functions of the different agents (Section 3.2). Then, we show that the optimal menu of contracts can be found by solving a joint surplus problem (Section 3.3).

3.2 Value Functions

3.2.1 Unemployed Worker's Problem

Unemployed workers consume a flow utility $b > 0$ while searching in the labor market. Search is directed toward the sub-market that offers the most profitable expected return for workers. In particular, unemployed workers look for employment in the labor market $i \in \mathcal{I}$ that yields the highest return and, within this market, they apply to the firm that offers the highest ex-ante payoff. Thus, the total value of unemployment is given by $\boldsymbol{U} = \max\{\boldsymbol{U}_{OE}, \boldsymbol{U}_{FT}\}$, where the value of searching for type-i contracts is defined by:

$$\boldsymbol{U}_i = \max_{W} U_i(W) \tag{5}$$

In turn, $U_i(W)$ solves the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho U_i(W) = b + \mu_i\big(\theta(W)\big) \max\Big(W - U_i(W), 0\Big) \tag{6}$$

Equation (6) states that the value of search is given by the leisure utility b plus the option value of matching with a firm promising continuation utility W. As workers prefer the most profitable offers, when unemployed they must remain indifferent ex-ante between all of those offers which they decide to apply to. This means that: (i) unemployed workers must be ex-ante indifferent between applying for an OE or an FT contract, and (ii) they must also be ex-ante indifferent between all firms making offers within the same contract type. Formally, condition (i) implies that $\boldsymbol{U}_{OE} = \boldsymbol{U}_{FT} = \boldsymbol{U}$. Requirement (ii), in turn, can be summarized by the following complementary slackness condition:

$$\forall (W, i) \in \mathbb{R}_+ \times \mathcal{I} : \quad U_i(W) \leq \boldsymbol{U}, \text{ with equality if, and only if, } \mu_i\big(\theta(W)\big) > 0$$

This condition states that submarkets either maximize the value of being unemployed, or are never visited by workers. Imposing this condition into (6) we find:[22]

$$\theta_i(W) = \mu_i^{-1}\left(\frac{\rho \boldsymbol{U} - b}{W - \boldsymbol{U}}\right) \tag{7}$$

This equation defines the equilibrium function that maps, for type-i contracts, promised values to market tightness, for any given value of unemployment $\boldsymbol{U}$. This mapping is used by both workers and firms to evaluate payoffs for any equilibrium in which unemployed workers earn an option value from searching for jobs (i.e. $\boldsymbol{U} \geq b/\rho$). In particular, note that market tightness is decreasing in W (more attractive contracts for workers ex-post attract more workers per job posting ex-ante), and increasing in $\boldsymbol{U}$ (a better outside option for workers makes job postings relatively less attractive ex-ante).[23]

3.2.2 Employed Worker's Problem

We now move to the problem of the employed worker. Assume this worker is performing a job of type $j \in \mathcal{J}$ and is employed by a firm of type $\varphi \in \Phi$ in state $(\vec{n}, z) \in \mathbb{N} \times \mathbb{Z}$ under contract $c_{ij} = \{w_{ij}, \delta_{ij}, p, W'_{ij}(\vec{n}', z')\}$. Throughout, we will make use of the short-hand notation $\vec{n}_{ij}^{+} \equiv (n_{ij} + 1, \vec{n}_{-(ij)})$, $\vec{n}_{ij}^{-} \equiv (n_{ij} - 1, \vec{n}_{-(ij)})$, $\vec{n}^{p} \equiv (n_{OEH}, n_{OEL} + 1, n_{FT} - 1)$, and $\vec{n}^{\tau} \equiv (n_{OEH} + 1, n_{OEL} - 1, n_{FT})$, to denote the various possible size transitions, where we define $\vec{n}_{-(ij)} \equiv \vec{n} \backslash \{n_{ij}\}$.[24] Given a menu of contracts $\mathcal{C} \equiv (c_{ij})_{(i,j) \in \mathcal{I} \times \mathcal{J}}$ currently paid by the firm, the worker's value satisfies the HJB equation:

$$\begin{aligned}
\rho \boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C}) \;&=\; w_{ij} + (\delta_{ij} + s_i^W + s^F)\Big(\boldsymbol{U} - \boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C})\Big) \\
\textit{Same type co-worker separates:} \quad & + (n_{ij} - 1)(\delta_{ij} + s_i^W)\Big(W'_{ij}(\vec{n}_{ij}^{-}, z) - \boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C})\Big) \\
\textit{Different type co-worker separates:} \quad & + \sum_{(i',j') \neq (i,j)} n_{i'j'}(\delta_{i'j'} + s_{i'}^W)\Big(W'_{ij}(\vec{n}_{i'j'}^{-}, z) - \boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C})\Big) \\
\textit{Firm hires with either contract:} \quad & + \sum_{i' \in \mathcal{I}} \eta_{i'}\Big(W'_{i'L}(\vec{n}_{i'L}^{+}, z)\Big)\Big(W'_{ij}(\vec{n}_{i'L}^{+}, z) - \boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C})\Big) \\
\textit{FT worker is promoted:} \quad & + n_{FT}\, p\Big(W_{ij}^{p}(\vec{n}^{p}, z) - \boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C})\Big) \\
\textit{Skill upgrade for OE low-types:} \quad & + n_{OEL}\tau\Big(W_{ij}^{\tau}(\vec{n}^{\tau}, z) - \boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C})\Big) \\
\textit{Productivity shock:} \quad & + \sum_{z' \in \mathbb{Z}} \lambda(z'|z)\Big(W'_{ij}(\vec{n}, z') - \boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C})\Big)
\end{aligned} \tag{8}$$

[22] In order to save on notation, throughout we will write $\theta(W)$ when we in fact mean $\theta(W, \boldsymbol{U})$. Nonetheless, the reader should keep in mind that $\boldsymbol{U}$ is an endogenous object that we solve for in equilibrium.

[23] More precisely, $\theta(W)$ is decreasing in W as long as $W \geq \boldsymbol{U}$. However, in equilibrium this condition will always hold thanks to a worker-participation constraint on the firm's problem.

[24] As $\vec{n}$ may contain more than one instance of the same element (e.g. the firm has five FE workers and five OEH workers), throughout the paper the symbols $\cup$ and $\backslash$ represent *multiset* union and difference operators, respectively, meaning $\{a, b\} \cup \{b\} = \{a, b, b\}$ instead of $\{a, b\} \cup \{b\} = \{a, b\}$, and $\{a, b, b\} \backslash \{b\} = \{a, b\}$ instead of $\{a, b, b\} \backslash \{b\} = \{a\}$.

The right-hand side of this equation incorporates the different sources of value for the employed worker. On the first line, the worker gets a wage w_{ij}, and the option value from separation into unemployment (second additive term), either because she gets laid off (at rate δ_{ij}), or her contract expires (at rate s_i^W), or because the firm exits (at rate s^F). The second and third lines include the change in value from the separation of a co-worker that was employed under the same contract and job type (second line), or under the other contract type, or job type, or both (third line). The fourth line is the change in value due to the firm hiring an additional worker for a low-type job with a contract of type i', where throughout we will use the short-hand notation $\eta\big(W(\cdot)\big) \equiv \eta\big(\theta(W(\cdot))\big)$ for the job-filling rate, with $\theta(\cdot)$ defined in equation (7). The fifth and sixth lines refer to the change in value when either a promotion or skill accumulation take place, where we have defined:

$$W^p_{ij}(\vec{n}^p, z) \equiv \begin{cases} \dfrac{1}{n_{FT}}\Big(W'_{OEL}(\vec{n}^p, z) + (n_{FT}-1)W'_{FT}(\vec{n}^p, z)\Big) & \text{if } i = FT,\ \forall j \\ W'_{OE,j}(\vec{n}^p, z) & \text{if } i = OE,\ \forall j \end{cases}$$

$$W^\tau_{ij}(\vec{n}^\tau, z) \equiv \begin{cases} W'_{FT}(\vec{n}^\tau, z) & \text{if } i = FT,\ \forall j \\ \dfrac{1}{n_{OEL}}\Big(W'_{OEH}(\vec{n}^\tau, z) + (n_{OEL}-1)W'_{OEL}(\vec{n}^\tau, z)\Big) & \text{if } (i,j) = (OE, L) \\ W'_{OEH}(\vec{n}^\tau, z) & \text{if } (i,j) = (OE, H) \end{cases}$$

The last line of equation (8) includes the change in value do a productivity shock.

3.2.3 Active Firm's Problem

Consider now an active firm of type $\varphi \in \Phi$ in state $(\vec{n}, z)$. This firm must choose a menu of contracts $\mathcal{C} = (c_{ij})$ with $c_{ij} = \{w_{ij}, \delta_{ij}, p, W'_{ij}(\vec{n}', z')\}$ for each $(i,j) \in \{FT, OE\} \times \{H, L\} = \mathcal{I} \times \mathcal{J}$, subject to the outstanding promises $\vec{W} = (W_{ij}, \vec{W}_{-(ij)})$ to its current workers, where we denote $\vec{W}_{-(ij)} \equiv \vec{W} \backslash \{W_{ij}\}$. Denote by $\boldsymbol{J}(\vec{n}, z, \varphi, \vec{W})$ the value of this firm. The HJB equation is:

$$\begin{aligned}
\rho \boldsymbol{J}(\vec{n}, z, \varphi, \vec{W}) \;=\; & \max_{\{w_{ij}, \delta_{ij}, p, W'_{ij}(\vec{n}', z')\}} \Bigg\{ Y(\vec{n}, z, \varphi) - \xi n_{FT} p^{\vartheta} + s^F \Big(\boldsymbol{J}^e - \boldsymbol{J}(\vec{n}, z, \varphi, \vec{W})\Big) \\
\textit{Wage bill and firing costs:} \quad & + \sum_{i \in \mathcal{I}} \Bigg[\sum_{j \in \mathcal{J}} \Bigg(- w_{ij} n_{ij} - \chi n_{ij} \delta_{ij}^{\psi} \\
\textit{Workers type } (i,j) \textit{ separate:} \quad & + n_{ij}(\delta_{ij} + s_i^W)\Big(\boldsymbol{J}\big(\vec{n}^-_{ij}, z, \varphi, \vec{W}'(\vec{n}^-_{ij}, z)\big) - \boldsymbol{J}(\vec{n}, z, \varphi, \vec{W})\Big)\Bigg) \\
\textit{Hiring under contract } i\textit{:} \quad & + \eta_i\Big(W'_{iL}(\vec{n}^+_{iL}, z)\Big)\Big(\boldsymbol{J}\big(\vec{n}^+_{iL}, z, \varphi, \vec{W}'(\vec{n}^+_{iL}, z)\big) - \boldsymbol{J}(\vec{n}, z, \varphi, \vec{W})\Big)\Bigg] \\
\textit{FT workers promoted:} \quad & + n_{FT} p \Big(\boldsymbol{J}\big(\vec{n}^p, z, \varphi, \vec{W}'(\vec{n}^p, z)\big) - \boldsymbol{J}(\vec{n}, z, \varphi, \vec{W})\Big) \\
\textit{Skill upgrade for OE low-types:} \quad & + n_{OEL} \tau \Big(\boldsymbol{J}\big(\vec{n}^\tau, z, \varphi, \vec{W}'(\vec{n}^\tau, z)\big) - \boldsymbol{J}(\vec{n}, z, \varphi, \vec{W})\Big)
\end{aligned}$$

$$\textit{Productivity shock:} \qquad + \sum_{z' \in \mathbb{Z}} \lambda(z'|z) \Big(\boldsymbol{J}\big(\vec{n}, z', \varphi, \vec{W}'(\vec{n}, z')\big) - \boldsymbol{J}\big(\vec{n}, z, \varphi, \vec{W}\big) \Big) \Bigg\} \tag{9}$$

This maximization problem is subject to two constraints:

$$\textit{Promise-keeping constraint:} \qquad \boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C}) \geq W_{ij}, \quad \forall (i, j) \tag{10a}$$

$$\textit{Worker-participation constraint:} \qquad W'_{ij}(\vec{n}', z') \geq \boldsymbol{U}, \qquad \forall (\vec{n}', z'),\ \forall (i, j). \tag{10b}$$

Equation (9) states that the firm chooses the menu of contracts that maximize firm value. Firm value is composed of the following terms: on the first line of equation (9), the firm makes profits from its current workers, net of wage, firing and promotion costs; the second line includes the changes in value due to firm exit (first term) and worker separation (second term), where $\boldsymbol{J}^e$ is the value of a potential entrant (derived below); the third line includes the event of the firm hiring a new worker; the fourth line is the event of a promotion of FT workers to an OE contract; the last line includes the change in value due to a productivity shock.

Importantly, this maximization problem is subject to two constraints. Constraint (10a) is a *promise-keeping* constraint: in choosing the contract, the firm must deliver an expected value to workers (left-hand side) which is no lower than the outstanding promise (right-hand side). This constraint is in place because of the firm's initial commitment to the posted contracts. Constraint (10b) is a *worker-participation* constraint: for every possible future state $(\vec{n}', z')$, the value that each worker obtains cannot be lower than its outside option. This constraint is in place because workers do not commit, and must therefore be enticed to remain matched.

3.2.4 Potential Entrant's Problem

Finally, we characterize the problem of the potential entrant firm. These are firms with no workers, perceiving a value $\boldsymbol{J}^e$. In order to post a contract, they must incur a flow cost κ and, upon entry, draw a permanent type $\varphi \in \Phi$ and an initial idiosyncratic productivity $z^e \in \mathbb{Z}$ from the π_φ and π_z distributions, respectively. Firms enter with one worker, whether it is under an OE or an FT contract. Formally, their problem is:

$$\rho \boldsymbol{J}^e = -\kappa + \sum_{\varphi \in \Phi} \sum_{z^e \in \mathbb{Z}} \sum_{i \in \mathcal{I}} \pi_\varphi(\varphi) \pi_z(z^e) \widetilde{\boldsymbol{J}}^e_i(z^e, \varphi) \tag{11}$$

where $\widetilde{\boldsymbol{J}}^e_i(z^e, \varphi)$ is the option value for a firm with productivity draws (z^e, φ) of entering with one worker employed under contract $i \in \mathcal{I}$, defined by:

$$\widetilde{\boldsymbol{J}}^e_i(z^e, \varphi) \equiv \max_W \left\{ \eta_i(W) \Big(\boldsymbol{J}\big(\vec{n}^e_{iL}, z^e, \varphi, \{W\}\big) - \boldsymbol{J}^e \Big) \quad \text{s.t. } W \geq \boldsymbol{U} \right\} \tag{12}$$

where $\boldsymbol{J}(\cdot)$ solves equation (9), and we have used the notation $\vec{n}^e_{ij} \equiv (n^e_{ij}, \vec{n}^e_{-(ij)}) = (1, \vec{0})$. The usual worker-participation constraint applies, $W \geq \boldsymbol{U}$. However, as the potential entrant does not yet have workers, there is no promise-keeping constraint. We assume free entry into the labor market, i.e. we allow the aggregate measure of firms F to freely adjust in equilibrium. This implies that, in an equilibrium with positive entry, we must have $\boldsymbol{J}^e = 0$.

3.3 Optimal Contract

We are now ready to characterize the optimal menu of contracts. As we have seen, these contracts maximize firm value subject to providing workers with enough utility (and, in particular, with at least the continuation value they were promised going forward when they first joined the firm). As it is standard in these type of models, in order to derive the optimal contract we can conveniently solve an equivalent and simpler problem, whereby the joint surplus of the match (that is, the sum of the firm's value and that of all of its workers) is maximized.

To show this equivalence, note first that by monotonicity of preferences, the promise-keeping constraint (10a) must always bind with equality in equilibrium. Otherwise, the firm could increase its value by offering a combination of flow and continuation payoffs to the workers that would yield lower value to them and still comply with the firm's initial promises. This means that, for a firm $(\vec{n}, z)$ promising values $\vec{W} = (W_{ij}, \vec{W}_{-(ij)})$ to each worker type, the value of its type-(i, j) workers $\boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C})$, which we defined in equation (8), must equal W_{ij}, the outstanding promise. In what follows, we will therefore write W_{ij} in place of $\boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \mathcal{C})$.

3.3.1 Joint Surplus Problem

Let us define the *joint surplus* of a match in firm of type φ in state $(\vec{n}, z, \vec{W})$ as the sum of the firm's value and the value of all of its workers:

$$\boldsymbol{\Sigma}(\vec{n}, z, \varphi) \; \equiv \; \boldsymbol{J}(\vec{n}, z, \varphi, \vec{W}) \; + \; \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} n_{ij} W_{ij} \tag{13}$$

As discussed in detail below, the joint surplus is independent from promised values and, anticipating this result, on the left-hand side of equation (13) we have written $\boldsymbol{\Sigma}(\vec{n}, z, \varphi)$ instead of $\boldsymbol{\Sigma}(\vec{n}, z, \varphi, \vec{W})$. In Appendix A.1 we show that $\boldsymbol{\Sigma}(\vec{n}, z, \varphi)$ solves the HJB equation:

$$\begin{aligned}
(\rho + s^F)\boldsymbol{\Sigma}(\vec{n}, z, \varphi) = \max_{p, \{\delta_{ij}, W'_{iL}(\vec{n}^+_{iL}, z)\}} & \Bigg\{ \boldsymbol{S}(\vec{n}, z, \varphi) + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} n_{ij}(\delta_{ij} + s^W_i) \Big(\boldsymbol{\Sigma}(\vec{n}^-_{ij}, z, \varphi) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi) \Big) \\
& + \sum_{i \in \mathcal{I}} \eta_i \Big(W'_{iL}(\vec{n}^+_{iL}, z) \Big) \Big(\boldsymbol{\Sigma}(\vec{n}^+_{iL}, z, \varphi) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi) \Big) + n_{FT}\, p \Big(\boldsymbol{\Sigma}(\vec{n}^p, z, \varphi) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi) \Big) \\
& + n_{OEL} \tau \, \Big(\boldsymbol{\Sigma}(\vec{n}^\tau, z, \varphi) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi) \Big) + \sum_{z'} \lambda(z'|z) \Big(\boldsymbol{\Sigma}(\vec{n}, z', \varphi) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi) \Big) \Bigg\},
\end{aligned} \tag{14}$$

subject to

$$W'_{iL}(\vec{n}^+_{iL}, z) \geq \boldsymbol{U}, \;\; \forall i \in \mathcal{I} \tag{15}$$

where we have used the short-hand notation:

$$\begin{aligned}\boldsymbol{S}(\vec{n},z,\varphi) \equiv\ & \underbrace{Y(\vec{n},z,\varphi)}_{\text{Firm's profits}} + \underbrace{\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} n_{ij}(\delta_{ij}+s_i^W+s^F)\boldsymbol{U}}_{\text{Workers' outside options}} \\ & - \underbrace{\xi n_{FT} p^{\vartheta}}_{\substack{\text{Promotion}\\ \text{costs}}} - \underbrace{\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} \chi n_{ij}\delta_{ij}^{\psi}}_{\text{Firing costs}} - \underbrace{\sum_{i\in\mathcal{I}} \eta_i\Big(W'_{iL}(\vec{n}^+_{iL},z)\Big)W'_{iL}(\vec{n}^+_{iL},z)}_{\text{Commitment costs}}\end{aligned} \tag{16}$$

to denote the expected flow surplus.[25] Equation (14) states that the joint surplus is composed of the flow surplus, plus the changes in joint surplus value due to worker separation or firing (first line of (14)), hiring or promotion (second line), and productivity shocks (third line).

Importantly, problem (14) is simpler than the firm's problem in equation (9) for two reasons. First, the space of payoff-relevant states has a lower dimension, namely $(\vec{n},z,\varphi)$ instead of $(\vec{n},z,\varphi,\vec{W})$. Second, the choice set is smaller as well: instead of having to choose a full contract vector $\mathcal{C} = \{w_{ij},\delta_{ij},p,W'_{ij}(\vec{n}',z')\}_{(i,j)\in\mathcal{I}\times\mathcal{J}}$, only the subset

$$\mathcal{C}_{\Sigma} \equiv \Big\{\delta_{ij},p,W'_{iL}(\vec{n}^+_{iL},z)\Big\}_{(i,j)\in\mathcal{I}\times\mathcal{J}} \subset \mathcal{C} \tag{17}$$

matters for the joint surplus. By contrast, the remaining elements of the contract set, i.e. $\widehat{\mathcal{C}} \equiv \mathcal{C}\backslash\mathcal{C}_{\Sigma} = \{w_{ij},W'_{ij}(\vec{n}^-_{ij},z),W'_{ij}(\vec{n}^p,z),W'_{ij}(\vec{n}^{\tau},z),\{W'_{ij}(\vec{n},z')\}_{z'\in\mathbb{Z}}\}_{(i,j)\in\mathcal{I}\times\mathcal{J}}$, leave the joint surplus unchanged, and are purely redistributive between the firm and its workers. Equipped with this formulation, we arrive at our main equivalence result:

Proposition 1 *The firm's and joint surplus problems are equivalent in the following sense:*

1. *For any set of contracts $\mathcal{C}$ that solves problem (9)-(10a)-(10b), the subset $\mathcal{C}_{\Sigma}\subset\mathcal{C}$ defined in (17) solves problem (14).*
2. *Conversely, if $\mathcal{C}_{\Sigma}$ is a solution to problem (14), then there exists a unique set of wages and continuation promises $\widehat{\mathcal{C}} \equiv \{w_{ij},W'_{ij}(\vec{n}^-_{ij},z),W'_{ij}(\vec{n}^p,z),W'_{ij}(\vec{n}^{\tau},z),\{W'_{ij}(\vec{n},z')\}_{z'\in\mathbb{Z}}\}_{(i,j)\in\mathcal{I}\times\mathcal{J}}$ such that $\widehat{\mathcal{C}}\cup\mathcal{C}_{\Sigma}$ solves problem (9)-(10a)-(10b).*

Proof. See Appendix A.1.

Proposition 1 establishes a useful equivalence result: in order to find the optimal contract, one may solve the joint surplus problem rather than the firm's problem. The proposition also lays a solution procedure, in two steps. In the first stage, we solve problem (14) to find the optimal job-filling rate $\eta_i(W'_{iL}(\vec{n}^+_{iL},z))$, firing rate δ_{ij}, and promotion rate p, for each job-contract pair. Since

[25] The flow surplus is the sum of the firm's flow profits and worker's outside options in case of separation (first line of (16)), net of three types of costs (second line, in this order): promotion costs, firing costs, and commitment costs (namely the costs of having to deliver the promised value in case of hiring, in expected terms).

the contract space is complete, all agents are risk-neutral, and utilities are fully transferable between them, there always exists a set of promised utilities $\vec{W}$ and wages $\{w_{ij}\}$ such that, for any future state of the match, rents can be redistributed across firm and workers in a surplus-maximizing way. Therefore, in the second stage, we find wages and promised utilities residually by ensuring that the promise-keeping constraint (10a) is binding and the worker-participation constraint (10b) is satisfied at all points of the state space.

As stated above, this two-step solution procedure may appear to lead to an indeterminacy problem between promised values and wages: indeed, from equation (8), there exist different combinations of these two objects that leave $\boldsymbol{W}_{ij}$ unchanged. This indeterminacy is resolved, however, by the free entry condition, which uniquely pins down the subset $\widehat{\mathcal{C}}$ via a simple iterative routine. In particular, the (unique) solution to the entrant's problem provides the outstanding promise of one-worker firms; from the optimal contract of these we may then find the outstanding promises of two-worker firms; and so on.

More precisely, we can write the free-entry condition in joint-surplus terms:[26]

$$\kappa = \max_{\{W_{iL}^e(z^e)\}} \left\{ \sum_{\varphi\in\Phi} \sum_{z^e\in\mathbb{Z}} \pi_\varphi(\varphi)\pi_z(z^e) \left[\sum_{i\in\mathcal{I}} \eta_i\Big(W_{iL}^e(z^e)\Big)\Big(\boldsymbol{\Sigma}(\vec{n}_{iL}^e, z^e, \varphi) - W_{iL}^e(z^e)\Big) \right] \right\} \tag{18}$$

The unique solution to this problem is the outstanding promised value of active firms with a single type-i worker and productivity z^e (given firm type φ) which, because of commitment, is precisely the state vector of these firms. For them, there is a unique downsizing decision $W_{iL}^-(\vec{n}_{iL}^e, z^e, \varphi)$, which is found from the free entry problem, and a unique promotion decision (in case $i = FT$), which is found as the solution of the joint-surplus maximization problem of type-φ firms with one FT-type worker and no OE workers. Using these, all that remains is to find the wage w_{iL}, but this can be found directly from the promise-keeping constraint (10a) binding with equality, i.e. such that $\boldsymbol{W}_{iL}(\vec{n}_{iL}^e, z^e, \varphi; \{c_{iL}\}) = W_{iL}^e(z^e)$, where $W_{iL}^e(z^e)$ is the solution to problem (18). This then gives a unique surplus-maximizing contract for the single-worker firm (working on a low-type job). The continuation promises in this contract are then taken as the outstanding promises of two-worker firms. Proceeding similarly through the $\vec{n}$ space, this procedure allows us to construct full sequences $\{c_{ij}(\vec{n}, z, \varphi)\}$ by forward iteration over $(n_{ij}, \vec{n}_{-(ij)})$, and for all $z \in \mathbb{Z}$.

3.3.2 Equilibrium Hiring, Firing and Promotion Policies

Hiring, firing, and promotion rates can be found by taking first-order conditions of problem (14). For the optimal upsizing choice $W_{iL}^+ \equiv W_{iL}'(\vec{n}_{iL}^+, z)$, we have:

$$\underbrace{\left.\frac{\partial \eta_i(W)}{\partial W}\right|_{W=W_{iL}^+} W_{iL}^+}_{\substack{\text{Promise delivered} \\ \text{to new worker}}} + \underbrace{\eta_i(W_{iL}^+)}_{\substack{\text{Wage adjustment} \\ \text{for pre-existing} \\ \text{workers}}} = \underbrace{\Big(\boldsymbol{\Sigma}(\vec{n}_{iL}^+, z, \varphi) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi)\Big) \left.\frac{\partial \eta_i(W)}{\partial W}\right|_{W=W_{iL}^+}}_{\text{Expected change in joint surplus from new worker}} \tag{19}$$

[26]To arrive at this expression, impose $\boldsymbol{J}^e = 0$ into equation (11).

Intuitively, firms equate the marginal cost of hiring, on the left-hand side, to the marginal benefit, on the right-hand side. Providing an additional util of W_{iL}^{+} yields costs both in terms of the new worker as well as the pre-existing ones. On the one hand, the firm must deliver its promise to the new worker in case she is hired, as captured by the first additive term on the left-hand side of (19). On the other hand, by providing this additional utility to the new worker, the promised utility of the pre-existing workers must be modified through the wage, as captured by the second additive term on the left-hand side of equation (19).[27] On the right-hand side, the marginal benefit is given by the increase in joint surplus from hiring, times the change in the probability of a hire.

Equation (19) determines the continuation promise of firms $(\vec{n}, z)$ after hiring an additional worker (on a low-type job), $W_{iL}'(\vec{n}_{iL}^{+}, z, \varphi)$, and therefore the starting promises of firms of type φ and size $\{\vec{n}_{iL}^{+}\}_{i\in\mathcal{I}}$. Starting from $(n_{OEH}, n_{OEL}, n_{FT}) = (0,0,0)$ via the free-entry condition, this allows us to construct the whole sequence of promised values in the $(\vec{n}, z, \varphi)$ space. In turn, the objects $W_{ij}'(\vec{n}_{ij}^{-}, z, \varphi)$, $W_{ij}'(\vec{n}^{p}, z, \varphi)$ and $W_{ij}'(\vec{n}^{\tau}, z, \varphi)$ are chosen to be consistent with this sequence.

For the firing and promotion rates, $\delta_{ij}(\vec{n}, z, \varphi)$ and $p(\vec{n}, z, \varphi)$, we have, respectively:

$$\delta_{ij}(\vec{n}, z, \varphi) = \left(\frac{\boldsymbol{\Sigma}\left(\vec{n}_{ij}^{-}, z, \varphi\right) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi) + \boldsymbol{U}}{\psi\chi} \right)^{\frac{1}{\psi-1}} \tag{20}$$

$$p(\vec{n}, z, \varphi) = \left(\frac{\boldsymbol{\Sigma}\left(\vec{n}^{p}, z, \varphi\right) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi)}{\vartheta\xi} \right)^{\frac{1}{\vartheta-1}} \tag{21}$$

Intuitively, firms equate the marginal firing and promotion costs to the marginal gain, given by the corresponding changes in the joint surplus value. Having found the optimal promises $\{W_{ij}'(\vec{n}, z, \varphi)\}$ and optimal rates $\{\delta_{ij}(\vec{n}, z, \varphi), p(\vec{n}, z, \varphi)\}$, it is then straightforward to find wages from the promise-keeping constraint (10a), which must bind with equality.

To close the characterization of the equilibrium, we must describe the distribution of firms and workers. The law of motion for the measure of firms $f_t(n_H, n_L, z, \varphi)$ is characterized by a set of flow equations, which we provide in full in Appendix A.2. Appendix A.3 then shows how to obtain the aggregate measure of active firms, $F \equiv \sum_{n_H} \sum_{n_L} \sum_{z} \sum_{\varphi} f(n_H, n_L, z, \varphi)$, as well as the aggregate unemployment rate, in the stationary equilibrium.

4 Estimation

In this section, we describe how we estimate the parameters of the model. We start by choosing functional forms for several objects (Section 4.1) and by describing our data (Section 4.2). Next, we separate the parameters into those whose values are set externally (Section 4.3) and those that are calibrated internally via the Simulated Method of Moments (Section 4.4). Finally, we explore some qualitative features, putting a special emphasis on the sources of misallocation (Section 4.5).

[27] One util of promised utility changes wages by the expected value, which is the probability that this util will have to be delivered (equal to the probability of hiring, $\eta_i(W_{iL}^{+})$).

4.1 Parameterization

Productivity process First, we must parameterize the productivity shock z. For this, we must choose the values $\{z_1, \ldots, z_k\}$, and the $k(k-1)$ intensity rates $\{\lambda(z'|z)\}$. As this is a potentially large number of parameters, we recover them from the discretization of a Ornstein-Uhlenbeck diffusion process for idiosyncratic productivity (in logs):

$$d\ln(z_t) = -\rho_z \ln(z_t)dt + \sigma_z dB_t \tag{22}$$

where B_t is a Wiener process, and (ρ_z, σ_z) are positive persistence and dispersion parameters.[28] Particularly, we recover the $\{\lambda(z'|z)\}$ intensity rates and $\{z_i\}_{i=1}^k$ productivity levels from discretizing this process using the Euler-Maruyama and Tauchen (1986) methods (details in Appendix C.2). For the entrant firms' productivity distribution π_z we take the ergodic distribution associated with the (calibrated) Markov chain implied by equation (22).

Matching function Second, we must choose a matching function. Following the literature, we choose a standard Cobb-Douglas specification:[29]

$$\mathcal{M}_i(f,u) = A_i f^\gamma u^{1-\gamma} \tag{23}$$

where, respectively, $A_i > 0$ and $\gamma \in (0,1)$ are the matching efficiency and elasticity. This implies meeting rates $\mu_i(\theta) = A_i\theta^\gamma$ for the worker, and $\eta_i(\theta) = A_i\theta^{\gamma-1}$ for the firm, in labor market $i = OE, FT$. The advantage of assuming a Cobb-Douglas functional form is that it leads to convenient analytical representations for the promised value and the job-filling rate. Using equation (19), some algebra shows:

$$W'_{iL}(\vec{n}^+_{iL}, z, \varphi) = \gamma \boldsymbol{U} + (1-\gamma)\Big(\boldsymbol{\Sigma}(\vec{n}^+_{iL}, z, \varphi) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi)\Big) \tag{24}$$

This expression is intuitive: the continuation promise $W'_{iL}(\vec{n}^+_{iL}, z)$ for a new worker under contract i is a weighted average of the worker's outside option, $\boldsymbol{U}$, and the marginal net joint surplus gain from the hire, $\boldsymbol{\Sigma}(\vec{n}^+_{iL}, z, \varphi) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi)$. The object $(1-\gamma)$ gives the share of the overall gains (net of the outside option) that accrue to the new worker. On the other hand, and using the definition of joint surplus (equation (13)), the firm obtains the following change in value from hiring:

$$\boldsymbol{J}\Big(\vec{n}^+_{iL}, z, \varphi, \vec{W}'(\vec{n}^+_{iL}, z, \varphi)\Big) - \boldsymbol{J}(\vec{n}, z, \varphi, \vec{W}) \;=\; \underbrace{\gamma\Big(\boldsymbol{\Sigma}(\vec{n}^+_{iL}, z, \varphi) - \boldsymbol{\Sigma}(\vec{n}, z, \varphi) - \boldsymbol{U}\Big)}_{\text{New surplus, shared with new hire}} \tag{25}$$

[28] An Ornstein-Uhlenbeck diffusion process is a type of mean-reverting process which can be thought of as the continuous-time analogue of an AR(1) process in discrete time.

[29] As the model is cast in continuous time, meeting rates $\mu_i(\theta) \equiv \mathcal{M}_i(\theta, 1)$ and $\eta_i(\theta) \equiv \mathcal{M}_i(1, \theta^{-1})$ need not take values in the unit interval, which gives us flexibility in the choice of a functional form.

$$+ \underbrace{\sum_{i' \in \mathcal{I}} \sum_{j \in \mathcal{J}} n_{i'j} \Big(W'_{i'j}(\vec{n}, z, \varphi) - W'_{i'j}(\vec{n}^{+}_{iL}, z, \varphi) \Big)}_{\text{Transfer of value between firm and pre-existing workers}}$$

The firm's marginal gain in value is composed of two terms. On the one hand, the firm absorbs the share γ of the total net gain in joint surplus that is not absorbed by the new hire.[30] On the other hand, since we assume that all workers within the firm that are employed under the same contract must earn the same value, there must be a transfer of value between the firm and all its pre-existing workers (for *both* contract types) after hiring takes place. This is captured by the second additive term. Therefore, in equilibrium, firms must strike a balance between the surplus they extract from new hires and the surplus they extract from their pre-existing workers when a hire takes place.

The optimal job-filling rate can then be written as follows:

$$\eta_i(\vec{n}^{+}_{iL}, z, \varphi) = A_i^{\frac{1}{\gamma}} \left[(1-\gamma) \frac{\mathbf{\Sigma}(\vec{n}^{+}_{iL}, z, \varphi) - \mathbf{\Sigma}(\vec{n}, z, \varphi) - \boldsymbol{U}}{\rho \boldsymbol{U} - b} \right]^{\frac{1-\gamma}{\gamma}} \tag{26}$$

Given our choice of $\gamma = 0.5$ (discussed in Section 4), equation (26) implies that the job-filling rate is linear in the ratio of the marginal net gain in joint surplus from a new match that accrues to a worker to the expected value of that worker's search. Therefore, in our calibrated model, firms can expect to find workers at a rate that is proportional to the returns that they offer to them.

Finally, with a Cobb-Douglas matching function the free-entry condition (equation (18)) reads:

$$\kappa = \gamma \left(\frac{1-\gamma}{\rho \boldsymbol{U} - b} \right)^{\frac{1-\gamma}{\gamma}} \sum_{\varphi \in \Phi} \sum_{z^e \in \mathbb{Z}} \pi_\varphi(\varphi) \pi_z(z^e) \sum_{i \in \mathcal{I}} \left[A_i \Big(\mathbf{\Sigma}(\vec{n}^{e}_{iL}, z^e, \varphi) - \boldsymbol{U} \Big) \right]^{\frac{1}{\gamma}} \tag{27}$$

Notice that the expected value of a single-worker firm (the right-hand side of this equation) is monotonically decreasing in the value of unemployment, because for higher $\boldsymbol{U}$ both the job-filling rate and the ex-post gains in match surplus that accrue to the firm are lower. In the numerical implementation of the model, we exploit this monotonic relationship to find the unique equilibrium value of unemployment, $\boldsymbol{U}$ (see Appendix C.1 for details).

4.2 Data Sources

First, we draw moments from the *Central de Balances* (CBI) firm-level data introduced in Section 2. We focus on a subsample of firms with at most 60 workers (see below), representing 97.3% of

[30] Indeed, by equation (24), this part of firm gains can be written as

$$\gamma \Big(\mathbf{\Sigma}(\vec{n}^{+}_{iL}, z, \varphi) - \mathbf{\Sigma}(\vec{n}, z, \varphi) - \boldsymbol{U} \Big) = \mathbf{\Sigma}(\vec{n}^{+}_{iL}, z, \varphi) - \mathbf{\Sigma}(\vec{n}, z, \varphi) - W'_{iL}(\vec{n}^{+}_{iL}, z, \varphi),$$

an identity stating that the share of net gains in joint surplus that is absorbed by the firm (left-hand side) must equal the total gains in joint surplus that are left after payments to the new hire (right-hand side).

firms in the full sample.[31] These data come at the yearly frequency. For our variables of interest (firm size, share of temporary workers, and value added per worker), we regress out the aggregate fixed effects (industry, time, province) to make sure that differences across firms do not reflect these other factors.

We also use aggregate data on worker flows into and out of employment by contract type from the *Encuesta de Población Activa* (EPA), the Spanish labor force survey. These data come at the quarterly frequency. Finally, to compute firm entry rates in the data, we use the *Directorio Central de Empresas* (DIRCE), a division within the Spanish national statistical agency (INE) collecting the number of firm births and deaths on a yearly basis.

4.3 Externally Calibrated Parameters

We set the model period to one quarter to match the EPA time frequency, as the CBI data (available at a yearly frequency) is used either for stock variables or for statistics that are stationary in the model. Note that, because the model is written in continuous time, we can accommodate any frequency observed in the data by means of the appropriate time aggregation in the model. To avoid a problem of state space dimensionality, we set $(N_{OEH}, N_{OEL}, N_{FT}) = (30, 15, 15)$, which imposes an upper bound of 60 workers per firm.[32]

Given the parameterization described in Section 4.1, we have 23 parameters to calibrate. Of these, eight parameters, namely $\mathbf{\Gamma}_{\text{ext}} \equiv \big(\rho, \gamma, \rho_z, \sigma_z, \psi, \vartheta, \tau, \zeta(\varphi_1)\big)$, are set externally (see Table 2). We fix the discount rate to $\rho = 0.0123$, corresponding to an annualized discount rate of $(1+\rho)^4 - 1 \approx 5\%$. For the matching elasticity, we choose $\gamma = 0.5$, a standard value in the literature (e.g. Petrongolo and Pissarides (2001)).[33] The productivity parameters (ρ_z, σ_z) introduced in equation (22) are calibrated to match a yearly autocorrelation of TFP of 0.81 and a yearly volatility of 0.34. We take these values from Ruiz-García (2021), which estimates an AR(1) process for firm-level TFP using Spanish firm-level balance sheet data from CBI, the same data source that we use in our empirical analysis. As we explain in Appendix C.2, these targets imply that $\rho_z = 0.0513$ and $\sigma_z = 0.1833$. We set the cost curvature parameters of the firing and promotion technologies to $\psi = \vartheta = 2$, so that layoff and promotion rates are linear in the corresponding net surplus changes, symmetrically to the hiring policy given that $\gamma = 0.5$ (recall equation (26)). We set the skill conversion rate to $\tau = 1/8$, such that the average duration before a skill upgrade is 8 quarters. This follows from Baley *et al.* (2023), who show that the returns to (occupational) experience are concave and almost exhausted after two years. Finally, we normalize the permanent productivity component of type-φ_1 firms to

[31] The empirical facts that we established in Section 2 for the full sample also hold for this sub-sample. For instance, we still find that the share of temporary workers increases (respectively, decreases) in firm size when looking at within-firm (respectively, between-firm) variation (see Figure D.2 in the Appendix).

[32] Given our choice of 5 productivity levels, 2 permanent firm types, and at most 60 employees per firm, the support of the state space has about 80,000 poins and the (sparse) transition matrix between states has 6bn potential transitions, making the calculation of the invariant distribution of firms computationally challenging. Appendix A.2 discusses this issue and how we get around it.

[33] This value is routinely used in models estimated to U.S. data, but has also been used for European labor markets, specifically in models of dual labor markets (e.g. Thomas (2006), Costain *et al.* (2010) and Bentolila *et al.* (2012)).

Table 2: Externally Calibrated Parameters

Parameter		Value	Target/Source
ρ	Discount rate	0.0123	5% annual discount rate
γ	Matching elasticity	0.5000	Petrongolo and Pissarides (2001)
ρ_z	Mean-reversion in productivity	0.0513	Ruiz-García (2021)
σ_z	Productivity dispersion	0.1833	Ruiz-García (2021)
ψ	Firing cost curvature	2.0000	Linear marginal gain from promoting
ϑ	Promotion cost curvature	2.0000	Linear marginal gain from firing
τ	Rate of skill upgrade	0.1250	Baley *et al.* (2023)
$\zeta(\varphi_1)$	Permanent productivity φ_1 firms	1.0000	Normalization (without loss)

$\zeta(\varphi_1) = 1$, which comes with no loss of generality as our economy exhibits size-neutrality —only the *relative* size $\zeta(\varphi_2)/\zeta(\varphi_1)$ matters, and will be calibrated internally.

4.4 Internally Calibrated Parameters

We have 15 parameters left: $\mathbf{\Gamma}_{\text{int}} \equiv \left(\kappa, \zeta(\varphi_2), \nu, \omega(\varphi_1), \alpha, \omega(\varphi_2), \chi, A_{OE}, A_{FT}, s^W_{OE}, s^W_{FT}, \xi, \pi_\varphi, s^F, b\right)$ to be calibrated internally. However, the joint estimation of the parameters and the assignment of firms to types φ is numerically unfeasible. We circumvent this problem by use of the two-step procedure in Bonhomme, Lamadon and Manresa (2022). We first assign individual firms to types by use of some statistics from the data without explicitly solving the model. Then, we estimate these model parameters by SMM conditional on this assignment. The parameters $\mathbf{\Gamma}_{\text{int}}$ are chosen to minimize the objective function:

$$\left(\vec{M}^{\text{data}} - \vec{M}^{\text{mod}}(\mathbf{\Gamma}_{\text{ext}}, \mathbf{\Gamma}_{\text{int}})\right)^{\top} \mathcal{W}^{-1} \left(\vec{M}^{\text{data}} - \vec{M}^{\text{mod}}(\mathbf{\Gamma}_{\text{ext}}, \mathbf{\Gamma}_{\text{int}})\right) \tag{28}$$

given $\mathbf{\Gamma}_{\text{ext}}$, where $\vec{M}^{\text{data}}$ is a vector of moments from the data (conditional on the assignment of firms to types), $\vec{M}^{\text{mod}}(\mathbf{\Gamma}_{\text{ext}}, \mathbf{\Gamma}_{\text{int}})$ is the model counterpart of these moments when the model is evaluated at parameters $(\mathbf{\Gamma}_{\text{ext}}, \mathbf{\Gamma}_{\text{int}})$, and $\mathcal{W}$ is a diagonal matrix of weights containing the squares of the data moments as the diagonal elements.

Due to the high degree of non-linearities in the model, an exact one-to-one identification of each parameter using a single moment from the data is impossible, since all model-generated moments are sensitive to changes in all parameters to varying degrees. Our calibration strategy then relies on picking moments from the data that are both (i) economically relevant for the mechanisms that underlie our key quantitative counterfactual exercises, and (ii) sufficiently sensitive to variation in each parameter (over a large enough support) relative to any other parameter. In the rest of this section, we argue that our calibration strategy satisfies criterion (i). Then, in Section 4.4.3, we show that criterion (ii) is satisfied as well. We do this by means of a formal (global) identification test that confirms the validity of our economic intuitions.

Table 3: Assignment of firms to permanent φ types

	Firm size (average, in # workers)	**Temporary share** (average, in %)	**Share of firms** (total, in %)
Firms type φ_1	9.8	4.3	19.7
Firms type φ_2	6.8	23.2	80.3
All	7.4	19.5	100.0

4.4.1 Assigning Firms to Permanent Types

The model predicts that firms of different technology types φ differ ex-post in their size, due to $\zeta(\varphi)$, and in their share of temporary workers conditional on firm size, due to $\omega(\varphi)$. In the data, we document a between-firm negative correlation between employment and temporary share (see Figure 1 for the whole sample and Appendix Figure D.2 for our calibration sample). Therefore, we classify firms into two different technology types with the aim of reproducing this relationship.

In particular, we proceed as follows. First, we regress the temporary share against dummies of firm size and unobserved firm fixed effects and we keep the estimated firm fixed effects. These can be thought of as capturing the "permanent temporary share" of each firm. Second, we take the time series average of firm size for each firm. This can be thought of as the "permanent size" of each firm. Third, we group all firms into 50 groups (2% of population each) based on the "permanent temporary share", from smallest to largest. Fourth, we compute the average "permanent temporary share" and the average "permanent size" in each group. Finally, we run a k-means algorithm with these two variables across all 50 groups to create only two groups, which are our two types (φ_1, φ_2).

The results of this procedure are in Table 3. We find that the first type of firms (type φ_1), which represent a smaller share of firms in the population (19.7%), make a moderate use of temporary contracts (4.3% of the firm's employment, on average). The remainder share of firms (80.3%), those of type φ_2, makes ample use of temporary contracts (23.2%). Additionally, the first group of firms achieves larger average firm size: 9.8 vs. 6.8 employees. In terms of the calibration, this may suggest that firm types that are more productive (larger $\zeta(\varphi)$) should also be more human capital intensive (larger $\omega(\varphi)$).

4.4.2 Targeted Moments

Using this split of our sample between types, we select various moments to identify our parameters. We group these moments in four blocks: (i) average and relative size; (ii) productivity and temporary share by firm characteristics; (iii) worker stocks and flows; and (iv) other moments.

Average and relative firm size First, we want the model to match the average firm size of each permanent type. To do so we target the average number of workers per firm and the relative size of firms of different types, which identify κ and $\zeta(\varphi_2)/\zeta(\varphi_1)$ respectively.

First, since there is a unit mass of workers by assumption, average firm size equals the ratio of the employment rate E to the measure of firms F. This is mostly affected by the entry cost parameter, κ. To see this, note that the employment rate E is pinned down by the combination of the unemployment-to-employment (UE), employment-to-unemployment (EU), and temporary-to-permanent employment (EE) worker flows, all of which are either direct calibration targets or implied by our calibration targets (see details below). Because κ monotonically lowers the firm's expected value upon entry through the free entry condition, it uniquely pins down the mass of active firms F and, therefore, the average firm size via E/F. Our target for average firm size is 7.4 employees (see Table 3).

Second, the *relative* firm size between φ_2 and φ_1 permanent firm types is driven by the ratio of their permanent productivity components, $\zeta(\varphi_2)/\zeta(\varphi_1)$. Because $\zeta(\varphi_1)$ is normalized to 1, this pins down $\zeta(\varphi_2)$. In the data, average firm size of φ_2 firms is a fraction 0.693 of the average firm size among φ_1 firms (Table 3). In the calibration, this delivers $\zeta(\varphi_2) = 0.8303$ (see Table 5, Panel A).

Productivity and temporary share by firm characteristics Next, we want the model to be consistent with the observed relationship between (i) firm productivity and firm characteristics, and (ii) temporary share and firm characteristics. These relationships should help identify the parameters in the production function, $\left(\nu, \alpha, \{\omega(\varphi)\}\right)$, and the costs of firing workers, χ.

To see this, let $n = n_H + n_L$ denote the total number of workers in a firm, our observable measure of size in the data. Using the production function (equation (2)), we can write log output per worker as:

$$\ln\left(\frac{Y(n_H, n-n_H, z, \varphi)}{n}\right) = z + \zeta(\varphi) - (1-\nu)\ln(n) + \frac{\nu}{\alpha}\ln\left(\omega(\varphi)\left(\frac{n_H}{n}\right)^{\alpha} + \left(1-\omega(\varphi)\right)\left(1-\frac{n_H}{n}\right)^{\alpha}\right). \tag{29}$$

If we could observe all the variables of this equation, a non-linear least squares regression could recover the degree of decreasing returns to scale ν from the partial effect of firm size on firm productivity. Moreover, we could recover the relative productivity of high and low skilled workers by firm type, $\omega(\varphi)$, and the elasticity of substitution between the two, $\frac{1}{1-\alpha}$, from the partial effect of changes in the skill composition of the firm on firm productivity. However, in our CBI data we observe neither firm productivity z nor the share of high-skilled workers, n_H/n.

To circumvent this issue, we follow an indirect inference strategy and consider a simplified version of the above equation that can be estimated in the data by OLS. In particular, we consider a second-order expansion of equation (29), we proxy output per worker with the ratio of value added (VA_{it}) to employment (Emp_{it}), replace the skill rate by the temporary share (as they are strongly correlated in the model but the former is unobserved), and we send the firm temporary TFP component z to the error term. The resulting regression is:

$$\ln\left(\frac{VA_{it}}{Emp_{it}}\right) = \text{constant} + \beta_0^A\,\mathbf{1}\left[\varphi_i = \varphi_2\right] + \beta_1^A\ln(Emp_{it}) + \beta_2^A\,TempSh_{it} + \beta_3^A\,TempSh_{it}^2 + \epsilon_{it}^A \tag{30}$$

for firm i at time t, where $\mathbf{1}[\varphi_i = \varphi_2]$ is an indicator variable for permanent firm type. We run this regression by OLS both in the model and the data, and we aim to match the regression coefficients β_1^A (which helps identify ν), and β_2^A and β_3^A (which help to jointly identify $\omega(\varphi_1)$ and α). Note that we pool firms of different types together but control for firm type φ by including the fixed effect $\mathbf{1}[\varphi_i = \varphi_2]$, which eliminates endogeneity concerns due to $\zeta(\varphi)$. In principle, both $\omega(\varphi_1)$ and $\omega(\varphi_2)$ matter for β_2^A and β_3^A. However, because φ_1 firms are much more prevalent in the data, this regression mostly helps identify $\omega(\varphi_1)$.

To identify $\omega(\varphi_2)$ and χ, we rely instead on variation in the temporary share. Particularly, we aim to match the effect of firm size and type on the temporary share. To do so, we follow again an indirect inference approach and run the following OLS regression both in the model and the data:

$$TempSh_{it} = \beta_0^B\, \mathbf{1}\big[\varphi_i = \varphi_2\big] + \sum_{\ell=1}^{N_{\text{bins}}} \beta_\ell^B\, \mathbf{1}\big[Emp_{it} \in \text{SizeBin}_\ell\big] + \epsilon_{it}^B \tag{31}$$

The β_0^B coefficient indicates the differential choice of temporary share by firms of different type (and same size), and should roughly pin down the ratio $\omega(\varphi_2)/\omega(\varphi_1)$, as this ratio drives the relative productivity across worker skills within a firm type, given firm size.[34] The $\{\beta_\ell^B\}$ coefficients, in turn, indicate the partial effect of firm size on the temporary share within a firm type, capturing the within-firm variation in Figure D.2.[35] This moment should inform about the firing cost shifter, χ, as this parameter drives the share of each worker type that a firm intends to keep in equilibrium.

Column A in Table 4 shows the empirical coefficients for regression (30). We see that conditional on type, larger firms are associated with slightly higher productivity ($\beta_1^A = 0.08$). This is the result of the positive correlation between firms size (n) and transitory firm productivity (z) in the cross-section, which beats the direct negative effect of decreasing returns to scale that we obtain in the calibration. We also see that, conditional on type, a larger temporary share is associated with a lower firm productivity ($\beta_2^A < 0$ and $\beta_3^A < 0$), consistent with the idea that low-skilled workers (which are more frequent among temporary workers) are less productive. Finally, we also find that low types are, on average, less productive ($\beta_0^A < 0$), which is not surprising given the results of our classification approach described in Table 3. Column B, in turn, shows the empirical coefficients of regression (31). The β_0^B coefficient equals 0.201 in the data, which says that, conditional on firm size and aggregate fixed effects, φ_2 firms (the less productive ones) use a higher fraction of temporary contracts, 20.1 percentage points more. On the other hand, the $\{\beta_\ell^B\}$ coefficients are monotonically increasing in ℓ, confirming that the within-firm positive relationship between firm size and temporary employment that we saw in Figures 1 and D.2 also holds when we condition on firm type instead of individual firm fixed effects.

[34] Indeed, taking the ratio of the marginal product of labor (MPL) of each type $j = H, L$ worker, we get $\frac{MPL_H}{MPL_L} = \frac{\omega(\varphi)}{1-\omega(\varphi)} \left(\frac{n_H}{n_L}\right)^{\alpha-1}$. A higher $\frac{\omega(\varphi_2)}{\omega(\varphi_1)}$ means therefore that firm type φ_2 should have a relatively stronger preference for H worker types, keeping (n_H, n_L) fixed. As H types cannot be employed under FT contracts, this parameter should drive the differential choices of temporary employment across firm types.

[35] In practice, we build size bins using 5-worker increments. As our sample contains firms with up to 60 employees, this implies $N_{\text{bins}} = 12$ size bins: 1-5, 6-10, 11-15, and so on up to 56-60.

Table 4: Regression evidence in the data

A. Productivity: $\ln(VA_{it}/Emp_{it})$			**B. Temporary share:** $TempSh_{it}$		
β_0^A	$\mathbf{1}[\varphi_i = \varphi_2]$	–0.003	β_0^B	$\mathbf{1}[\varphi_i = \varphi_2]$	0.201
β_1^A	$\ln(Emp_{it})$	0.081	β_1^B	$Emp_{it} \in [1, 5)$	0.005
β_2^A	$TempSh_{it}$	–0.104	β_2^B	$Emp_{it} \in [5, 10)$	0.037
β_3^A	$TempSh_{it}^2$	–0.092	β_3^B	$Emp_{it} \in [10, 15)$	0.080
			β_4^B	$Emp_{it} \in [15, 20)$	0.099
			β_5^B	$Emp_{it} \in [20, 25)$	0.113
			β_6^B	$Emp_{it} \in [25, 30)$	0.122
			β_7^B	$Emp_{it} \in [30, 35)$	0.129
			β_8^B	$Emp_{it} \in [35, 40)$	0.135
			β_9^B	$Emp_{it} \in [40, 45)$	0.142
			β_{10}^B	$Emp_{it} \in [45, 50)$	0.150
			β_{11}^B	$Emp_{it} \in [50, 55]$	0.156
			β_{12}^B	$Emp_{it} \in [55, 60]$	0.190
# observations		6,316,320	# observations		6,664,229
R^2		0.01	R^2		0.14

Note: Variables are all net of aggregate FE (sector, region, and province). All coefficients are significant at the 1% level.

In Panel B of Table 5, we report the calibrated model's fit on this regression evidence. In our indirect inference approach, we obtain $\nu = 0.7031$, showing strong decreasing returns to scale. Because temporary contracts are more prevalent among less skilled workers, this recovers $\omega(\varphi_1) = 0.7720 > 0.5$ and $\omega(\varphi_2) = 0.4534 < 0.5$. Moreover, we find $\alpha = 0.4840$, which delivers a considerably high elasticity of substitution between worker skill types, $\frac{1}{1-\alpha} = 1.9380$, but overall complementarity between skill types ($\alpha < \nu$). Then, we pin down a value for χ using $\beta_2^B - \beta_1^B$ (equal to 0.032 in the data), which captures the intensity by which the share of temporary workers changes within firm type, on average, when firms transition from the first to the second size bin. We choose to match the gap between these coefficients because the first two size bins comprise the vast majority of firms in our data. Given these parameter values, the calibrated model matches all of these coefficients quite well. In particular, the model is able to reproduce the *within-firm* relationships between productivity and the temporary share and between the temporary share and firm size. The *between-firm* differences in the temporary share are also well-captured: in the data, φ_1 and φ_2 firms have average temporary shares of 4.3% and 23.2% respectively; in the calibrated model, these numbers are 4.6% and 27.6%.

Worker stocks and flows Next, to capture employment dynamics, we target the employment-to-unemployment (EU) and unemployment-to-employment (UE) quarterly flow rates by contract type from the Spanish labor force survey (EPA), plus the average rate of temporary employment.

Appendix A.4 explains how we compute these worker flow rates both in the data and in the model. For the model-implied rates, we construct a simple 4-state Markov model of employment flows. The four Markov states are unemployment, employment in a FT contract, and employment in an OE contract with either low of high human capital (full details in Appendix A.4). In this model, the employment flow matrix has seven independent flows: UE flows for FT contracts and for OE contracts experienced by workers of low skill; EE flows from FT to OE contracts experienced by workers of low skill, and from low to high skill experienced by OE workers; and EU for FT and OE contracts alike, irrespective of human capital. The system implies three steady-state ratios of stocks: the unemployment rate, the share of temporary employment, and the share of high-skill workers (henceforth, the "skill share").

In the data, however, we do not observe skill levels, only contract types, so we target the overall quarterly flow rates for each contract type by adding up flows across skill levels in the model. Moreover, in order to make empirical flows fully consistent with the model's assumptions, we assume that the EE flow from OE to FT contracts is equal to zero. This leaves us with five flow rates: UE and EU rates for each contract type, and EE rates from FT to OE (which we call the "promotion rate"). These flow rates pin down two steady-state ratios: the temporary share and the unemployment rate. In the calibration, we target four of these five flows (leaving the promotion rate free) and the average temporary share (leaving the unemployment rate free).

Which parameters are informed by these five targets? First, the two UE rates are primarily affected by the matching efficiency parameters (A_{OE}, A_{FT}), as these act as shifters for the number of matches that take place in each market per unit time, given market tightness. The calibration delivers $A_{FT} = 1.9612$ and $A_{OE} = 0.2029$. In the data, UE transitions are far more frequent among FT workers (18.5% quarterly) than among OE workers (2.7%). The model rationalizes this by making the FT labor market a lot more "liquid": for given market tightness, there are more matches in the FT market per unit of time for every match that takes place in the OE market.[36] It should be pointed out, however, that these large differences in matching efficiency are only a quantitative feature of the calibration (allowing us to match the huge gap in UE rates across contract types that we see in the data) but are not a foregone conclusion of our theory.[37]

Secondly, the EU rates are most directly affected by the worker exogenous separation rates,

[36] In Spain during our period of analysis, on average more than 90% of the contracts signed each month were fixed-term (see Felgueroso, García-Pérez, Jansen and Troncoso-Ponce (2018)), indicative that this is indeed a very liquid market. Other studies, e.g. de Graaf-Zijl, van den Berg and Heyma (2011), argue that temporary contracts may ease job finding.

[37] Indeed, using equation (26), some algebra shows that the job-filling rates across contracts satisfy:

$$\underbrace{\frac{\eta_{FT}(\vec{n}^{+}_{FT}, z, \varphi)}{\eta_{OE}(\vec{n}^{+}_{OEL}, z, \varphi)}}_{\text{Gap in job-filling rates}} = \underbrace{\left(\frac{A_{FT}}{A_{OE}}\right)^{\frac{1}{\gamma}}}_{\substack{\text{Matching} \\ \text{efficiency gap}}} \underbrace{\left(\frac{W'_{FT}(\vec{n}^{+}_{FT}, z, \varphi) - \boldsymbol{U}}{W'_{OEL}(\vec{n}^{+}_{OEL}, z, \varphi) - \boldsymbol{U}}\right)^{\frac{1-\gamma}{\gamma}}}_{\text{Gap in net promised values}}$$

Thus, $\eta_{FT} > \eta_{OE}$ follows from the gap in matching efficiency and the gap in promised values. When $A_{FT} = A_{OE}$, we can still obtain $\eta_{FT} > \eta_{OE}$, as in the data, if for example $W'_{FT} > W'_{OE}$ for a sufficiently populated subset of the state space. For instance, this can be the case when φ_2 firms have a stronger need for FT workers (e.g. in cases when $\omega(\varphi_2) < 0.5$, as in our calibration), and there are sufficiently many such firms in the economy.

(s^W_{OE}, s^W_{FT}). In particular, in the data, transitions into unemployment are far more common in the FT market (12.97% quarterly) than they are in the OE market (1.39%). The calibration delivers that the separation rate of FT workers is much higher (equal to 0.5981, i.e. an average duration of about 5 months on the job, conditional on no endogenous separations) than that of OE workers (equal to 0.0429, i.e. an average duration of nearly 6 years on the job, conditional on no endogenous separations). We interpret the exogenous separations in OE contracts as voluntary quits of workers, while the exogenous separations in FT contracts may be driven by voluntary quits but also by regulations on the maximum duration of these contracts. The latter will be our preferred interpretation in our policy analysis of Section 5.

Finally, the promotion cost parameter ξ determines the flow of workers being promoted from FT to OE contracts. If we were to target this flow, then together with the other four flows described above we would uniquely pin down the unemployment rate and the temporary share. To identify ξ, we may therefore target the steady state temporary share in the EPA data of 21.9%.[38]

Other moments Finally, it remains to pin down values for the probability of entering as a given type, say φ_1; the firm exit rate, s^F; and the value of leisure, b. First, $\pi_\varphi(\varphi_1)$ is pinned down by the share of firms of this type among all active firms in the stationary solution of the model, targeted to be 19.7% (see Table 3).[39] In the calibration, we obtain $\pi_\varphi(\varphi_1) = 0.0667$, so that 6.7% of entering firms enter as the high type. Notice that there is a substantial difference between the share of φ_1-type firms in the stationary distribution vis-à-vis in the pool of entrants, indicating higher survival probabilities for high-type firms.

Second, to pin down s^F we target the entry rate of firms. In the data, we take this number from García-Perea, Lacuesta and Roldan-Blanco (2021), who in turn use data from DIRCE. The time-series average in the firm entry rate for the period 2004-2019 is around 8.6% annually. In the model, the inflow of firms equals its outflow, so we compute the firm entry rate as the ratio of actual entrants to the total measure of active firms.[40] This gives $s^F = 0.0204$, predicting a annual entry rate of 8.6%, the same as in the data (see Table 5).

The last remaining parameter is the opportunity cost of employment for the worker, b, which we calibrate to match that the income flow from unemployment represents 70% of the average worker productivity, a customary target in the literature (e.g. Hall and Milgrom (2008)).

4.4.3 Global Identification Results

To validate the identification of the 15 internally estimated parameters, we run the following exercise. For each parameter-moment pair established in the text (see Table 5 for the summary), we

[38] Recall that the EPA numbers are always slightly higher than the firm-level ones (see Figure D.1). We choose the EPA number to be consistent with the fact that our targets for the worker flow rates have been computed using this database.

[39] In the model, we compute F_{φ_1}/F, where $F_{\varphi_1} \equiv \sum_i \sum_j \sum_z f\left(\{n_{ij}\}, z, \varphi_1\right)$, and $f\left(\{n_{ij}\}, z, \varphi_1\right)$ is the stationary measure of type-φ_1 firms of productivity z with n_{ij} workers of type $j = L, H$ in contracts of type $i = FT, OE$.

[40] That is, $\frac{F_e}{F}\left[\sum_{\varphi\in\Phi}\sum_{z^e\in\mathbb{Z}}\pi_\varphi(\varphi)\pi_z(z^e)\sum_{i\in\mathcal{I}}\eta_i\left(W'_{iL}(\vec{n}^e_{iL}, z^e, \varphi)\right)\right]$.

Table 5: Internally Estimated Parameters and Model Fit

Parameter		**Value**	**Indentifying Moment**	**Model**	**Data**	**Source**
			Panel A. *Absolute and relative firm size*			
κ	Fixed firm entry cost	4.29×10^4	Average firm size	7.1	7.4	CBI
$\zeta(\varphi_2)$	Permanent productivity φ_2 firms	0.8303	Relative size φ_2 firms	0.612	0.693	CBI
			Panel B. *Productivity and temporary share by firm characteristics*			
ν	RTS parameter	0.7031	β_1^A coefficient, eq. (30)	0.105	0.081	CBI
$\omega(\varphi_1)$	Productivity H workers φ_1 firms	0.7720	β_2^A coefficient, eq. (30)	–0.087	–0.104	CBI
α	Substitutability worker types	0.4840	β_3^A coefficient, eq. (30)	–0.118	–0.092	CBI
$\omega(\varphi_2)$	Productivity H workers φ_2 firms	0.4534	β_0^B coefficient, eq. (31)	0.249	0.201	CBI
χ	Firing cost shifter	0.8789	$(\beta_2^B - \beta_1^B)$ gap, eq. (31)	0.035	0.032	CBI
			Panel C. *Worker stocks and flows*			
A_{OE}	Matching efficiency (OE market)	0.2029	UE rate, OE (quarterly)	0.011	0.027	EPA
A_{FT}	Matching efficiency (FT market)	1.9612	UE rate, FT (quarterly)	0.165	0.185	EPA
s_{OE}^W	OE Contract destruction rate	0.0429	EU rate, OE (quarterly)	0.019	0.014	EPA
s_{FT}^W	FT Contract destruction rate	0.5981	EU rate, FT (quarterly)	0.153	0.130	EPA
ξ	Promotion cost shifter	0.4339	Temporary share	0.247	0.219	EPA
			Panel D. *Other moments*			
$\pi_\varphi(\varphi_1)$	Probability of entering φ_1	0.0667	Share of φ_2-type firms	0.130	0.197	CBI
s^F	Firm destruction rate	0.0204	Firm entry rate (annual)	0.086	0.086	INE
b	Employment opportunity cost	0.6218	Leisure to output p.w.	0.650	0.700	.

Notes: The model period is one quarter. The table reports the values of the parameters estimated internally by minimizing the criterion function in equation (28), and their corresponding moments (see Section 4.4 for a description). UE and EU rates are averages over HP-filtered quarterly series from the EPA over the period 2005Q1-2018Q4 (data before 2005 is unavailable). *Data sources:* "CBI" means our subsample from the Central de Balances Integrada data; "INE" means data from the Instituto Nacional de Estadística; "EPA" means data from the Encuesta de Población Activa.

allow for quasi-random variation in all remaining parameters and solve the model for each such parameter configuration.[41] As a result, for each level of the identified parameter we obtain a whole distribution for the targeted moment. Then we plot, for each parameter-moment pair, the median of this distribution (black dots) and the inter-quartile range (shaded area). For reference, we also show the calibrated parameter value (dashed red vertical line), and the data and model moment values (dashed black and solid blue lines).

We then say that a parameter is well-identified by the moment when (i) the distribution changes across different values of the parameter; (ii) the rate of this change is high; (iii) the inter-quartile range of the moment's distribution is narrow throughout the support for the parameter; (iv) at the

[41] More precisely, we use the following procedure. First, we set wide enough bounds for each parameter from the $\mathbf{\Gamma}_{\text{int}}$ parameter vector. Then, we pick quasi-random realizations from the resulting hypercube using a Sobol sequence, which successively forms finer uniform partitions of the parameter space, comprehensively and efficiently exploring every corner of it. Finally, for each parameter combination, we solve the model and store the relevant moments. To implement this procedure, we use a high-performance computer, which allows us to parallelize the procedure, saving us a huge amount of computational time. In total, we solve the model 5 million times.

calibrated parameter value, the empirical target falls within the inter-quartile range.[42] Because all the remaining parameters are not fixed but instead are varying in a quasi-random fashion within a wide support, this method gives us a global view of identification. Figure D.3 in the Appendix presents the results from this identification procedure. We find that all parameters are well-identified by their corresponding moments along most criteria.

4.5 Misallocation

Before proceeding to our counterfactual exercises, we study the extent of worker misallocation generated by the search and matching frictions and the dual labor market structure. To do so, it is useful to have a first-best benchmark allocation. Appendix B derives the first-best allocation of workers that would be chosen by a planner that (i) is not constrained by the search frictions of the market economy, but (ii) takes the allocation of firms across productivity types as given. Using this benchmark allows us to measure the extent of misallocation in the assignment of employment (both regarding the overall size of firms as well as the skill composition of their workforce) that results from the search frictions in the competitive equilibrium (CE).

In particular, for a given distribution of firms across productivity types, there is no dispersion in the marginal product of each type of labor across firms in the first best allocation. More specifically, Appendix B shows that more productive firms employ more workers (due to decreasing retuns to scale), with the optimal allocation of worker skills to firms satisfying:[43]

$$\frac{n_H^*(z,\varphi)}{n_L^*(z,\varphi)} = \left(\frac{\omega(\varphi)}{1-\omega(\varphi)}\right)^{\frac{1}{1-\alpha}} \tag{32}$$

Equation (32) says that the planner seeks positive assortative matching between firms and workers in the following sense: the assignment of high-skilled workers should be directed toward the permanent firm type φ where these workers are most productive, as measured by the relative productivity of high-skilled vis-à-vis low-skilled workers that are employed by a φ-type firm, adjusted by the elasticity of substitution between the two, $\frac{1}{1-\alpha}$. Moreover, within each permanent productivity type, this assignment is independent from idiosyncratic productivity z, as z's affect high and low

[42] Criterion (i) implies that the moment is globally sensitive to variation in the parameter; (ii) gives an idea of how strong this sensitivity is; (iii) measures how much *other* parameters matter to explain variation in the moment; (iv) implies that the empirical target is not an outlier occurrence at the calibrated value of the parameter.

[43] To derive equation (32), we assume that the planner can choose the aggregate measures of high and low skilled workers, E_H and E_L. Alternatively, in the Appendix we also derive the optimal allocation of worker types to firm productivity types when the planner takes these measures as given. In that case, the optimal allocation of worker types to a given firm of productivity (z,φ) holds:

$$\frac{n_H^*(z,\varphi)}{n_L^*(z,\varphi)} = \frac{\widehat{\omega}(\varphi)}{\frac{1}{F}\sum_{\varphi'\in\Phi}\widehat{\omega}(\varphi')F_{\varphi'}}\left(\frac{E_H}{E_L}\right)$$

where $\widehat{\omega}(\varphi) \equiv \left(\frac{\omega(\varphi)}{1-\omega(\varphi)}\right)^{\frac{1}{1-\alpha}}$, and F_φ is the measure of firms of type φ. So, once again, the optimal skill share is constant in idiosyncratic productivity, z, and skewed toward the firm type where high-skilled types are relatively more productive. For this planner's solution, however, output losses from misallocation are typically smaller.

skilled workers equally. As a consequence, the socially-optimal skill share of any firm, $h^*(\varphi) \equiv \frac{n_H^*(z,\varphi)}{n_H^*(z,\varphi)+n_L^*(z,\varphi)}$, is also only a function of its permanent type, φ.

The decentralized economy differs from the efficient allocation of employment in two ways. First, within firm productivity type (z, φ), firms in the CE display different amounts of workers and skills. Second, the average amount of workers and skills allocated across productivity types (z, φ) in the CE differs from what the planner would choose. Both of these translate into output losses. Quantitatively, we find that our calibrated economy produces 6.8% less output per worker than that of the planner economy with the same level of employment and same distribution of firms, showing that search and matching frictions in Spain are very damaging for welfare.[44] Of this output loss, 53.3% is due to the between productivity type (z, φ) misallocation, and the other half is due to the within productivity type (z, φ) component.[45]

Figure 2 shows the extent of misallocation of workers across firm productivity types (z, φ). The top two panels describe the allocation of the total number of workers across firm types φ (left vs. right panels) and transitory productivity states z (different crosses within each panel) for the competitive economy (red schedule) and the planner economy (black schedule). We see that the competitive economy allocates too few workers to the most productive firms: among the high types (φ_1), the market allocates an average of 26.6 workers to the highest productivity shocks (z_5), while the planner would allocate 59.9 workers. This misallocation reflects the mean-reversion of productivity shocks and the rigidity of OE contracts. It is efficient for very productive type-φ_1 firms to employ many high-skilled workers. However, the good shocks may turn into bad shocks in the future, which in the CE economy would make it very hard for firms to get rid of redundant workers hired under OE contracts.

The bottom two panels in Figure 2 describe the allocation of human capital (the ratio of high skilled workers to all workers) across firms. As discussed above, the planner's allocation is invariant in z and increases in φ. The competitive equilibrium fails to allocate the right amount of human capital across firms. In particular, it does allocate more human capital to type-φ_1 firms than to type-φ_2 firms, but the allocation of human capital to type-φ_1 firms is way below the efficient one, especially at the top (high z), while type-φ_2 firms get a similar amount of human capital in both the

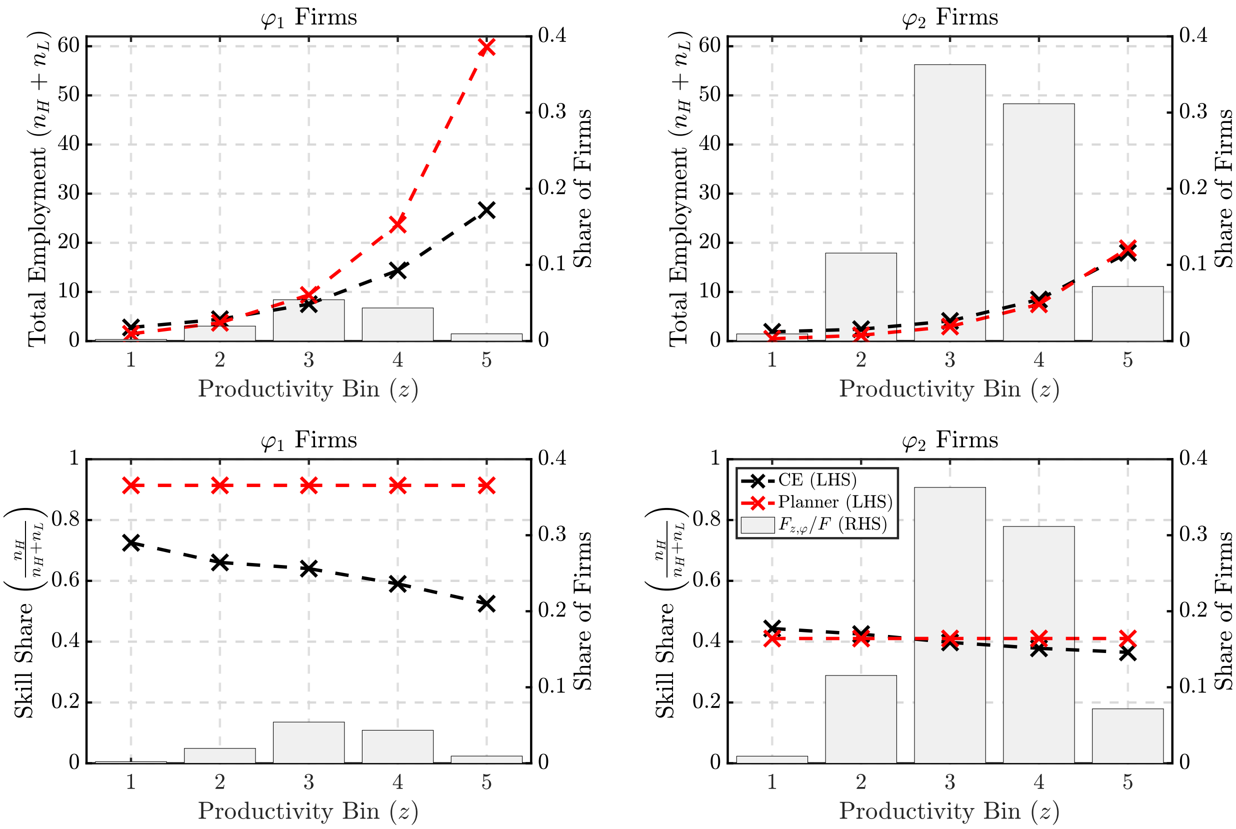
[44]For the planner's aggregate output, $\mathcal{Y}^*$, we use that equation (2) is homogeneous of degree ν to compute

$$\mathcal{Y}^* \equiv \sum_z \sum_\varphi F_{z,\varphi} \left\{ e^{z+\zeta(\varphi)} \left(n^*(z,\varphi)\right)^\nu \left[\omega(\varphi)\left(h^*(\varphi)\right)^\alpha + (1-\omega(\varphi))\left(1-h^*(\varphi)\right)^\alpha\right]^{\nu/\alpha} \right\}$$

where $n^*(z,\varphi)$ is given by equation (B.4) and plotted in the top panels of Figure 2, and $h^*(\varphi) = \left(1 + \left(\frac{\omega(\varphi)}{1-\omega(\varphi)}\right)^{\frac{1}{\alpha-1}}\right)^{-1}$ is the optimal skill share, plotted in the bottom panels of Figure 2. For the aggregate output in the CE, $\mathcal{Y}$, we use equation (33) introduced in the next section. Then, we find $\mathcal{Y}$ is 6.8% lower than $\mathcal{Y}^*$.

[45]To quantify the contribution of each margin, we construct a counterfactual aggregate output that aggregates across productivity levels and ignores dispersion in employment within each productivity group, so that all output losses are due to the between-productivity component when comparing this output to the social planner's. To be exact, $\widehat{\mathcal{Y}}^{CE}$, is computed using the same formula as $\mathcal{Y}^*$, except replacing $n^*(z,\varphi)$ by $n^{CE}(z,\varphi)$, the average employment within a (z,φ) group in the CE, and $h^*(\varphi)$ by $h^{CE}(z,\varphi)$, the average skill share within a (z,φ) group in the CE. This counterfactual output is 3.6% lower than the planner's. Therefore, in the calibrated economy, about half ($3.6/6.8 = 53.3\%$) of the output loss is due to the between component and the other half is due to the within-firm component.

Figure 2: Misallocation Across Productivity Classes in the Baseline Calibration.



Notes: This figure compares the allocation of employment (top row) and of the skill share (bottom row) for firms of different z's (x-axis) and φ's (left and right columns), in the competitive equilibrium (black schedule) and the social planner's (red schedule) solutions. For the competitive equilibrium (CE), we provide the average employment within each productivity group. For the planner's case, we provide the optimal employment level of the corresponding productivity group, using result (32). Full details can be found in Appendix B. The magnitudes corresponding to these two cases can be found on the left-hand axes. We also plot the stationary distribution of firms across productivity levels in the CE (right-hand axes).

CE and planner economies. In numbers, the planner assigns 91.4% of workers to be high-skilled in φ_1 firms, while the market allocates only 75% in the bottom (z_1) and 52.5% in the top (z_5) firms. This uncovers and important inefficiency of the menu of contracts available in the dual labor market economy. In particular, OE contacts are too rigid for type-φ_1 firms. For these firms, it would be efficient to face less worker turnover such that their employees have time to acquire more human capital. The tool to achieve this worker stability would be to employ more workers under OE contracts and less workers under FT contracts. But OE workers are too rigid and expensive to undo whenever bad productivity shock arise, which leads to an inefficient provision of human capital.

5 The Macroeconomic Implications of Dual Labor Markets

What are the micro and macroeconomic effects of dual labor markets, and what impact may government policies that regulate fixed-term contracts have on misallocation, aggregate productivity, and unemployment? In this section we use our calibrated model to study the macro effect of changes in the properties of FT contracts through their impact on worker and firm flows, and the composition of firms and skills in the economy.

To tackle this question, we solve for a series of economies in which we vary the exogenous separation rate for FT contracts, s_{FT}^{W}, such that average duration for FT contracts moves up from one month to a year (the average duration is 1.67 quarters, or about 5 months, in the calibrated economy). We leave all other parameters unchanged at their calibrated values, and compare across steady-state solutions.

5.1 A First Look: Macro Aggregates

The results for this series of policy exercises on a collection of relevant macroeconomic aggregates are reported in Figure 3, while Table 6 provides exact numbers for the two policies at the extreme (one with one month duration and another one with one year duration).

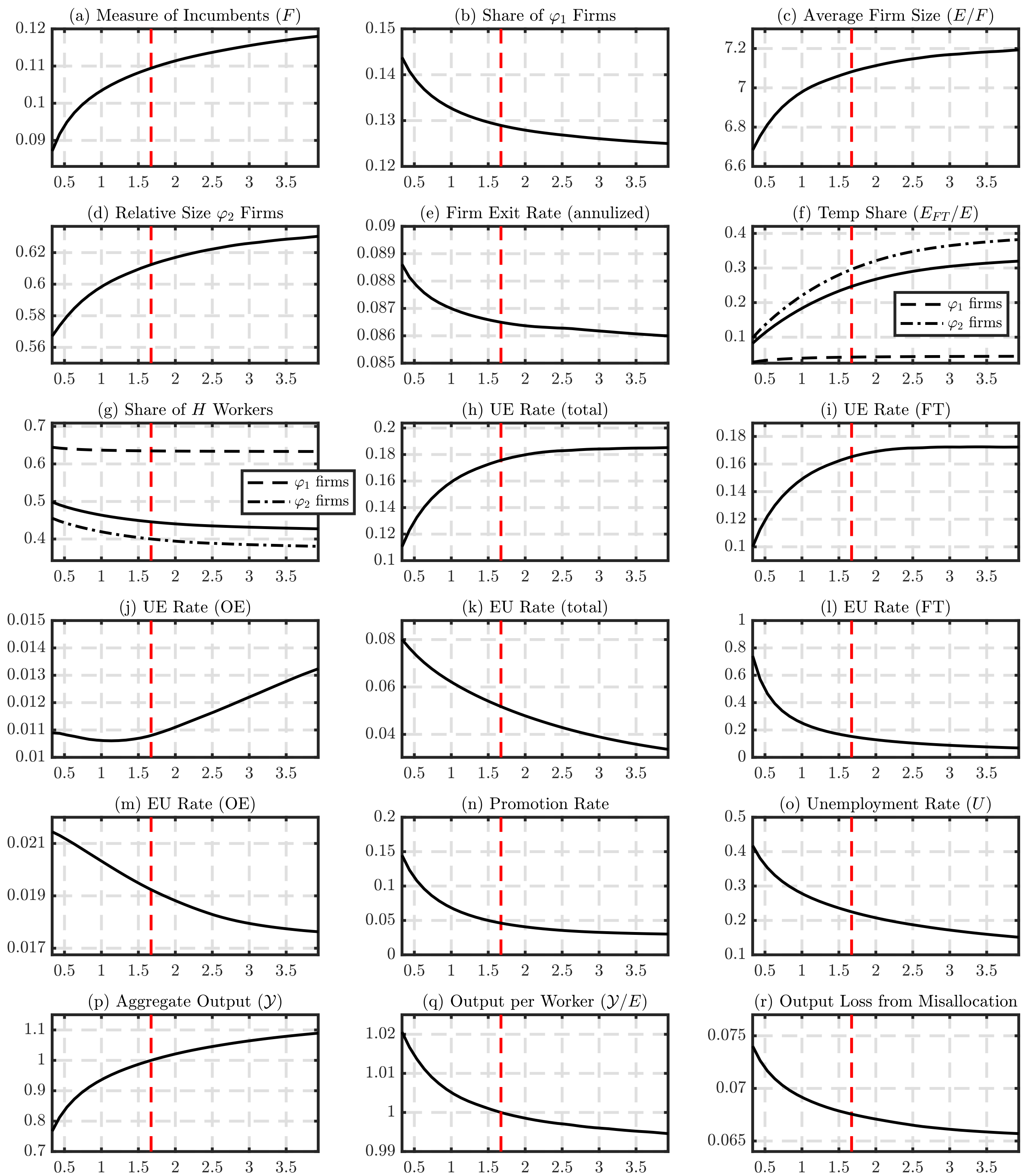
The results reveal a number of level and composition effects. Our main result is that a policy that reduces the legal duration of FT contracts from the baseline duration down to, say, one month (Column (A) in Table 6 and left-most point in each panel of Figure 3) has the intended effect of reducing the overall temporary share of the economy (see Panel (f) of Figure 3) by 66.6%, and increasing aggregate productivity (Panel (q)) by 2.1%. However, this policy would come at the expense of a large reduction in output (Panel (p)) via an increase in the unemployment rate (Panel (o)) that is larger than the increase in aggregate productivity.

5.1.1 Aggregate Employment and its Composition

In the calibrated model, firms face a trade-off between the opportunity costs of having FT workers due to high worker turnover and the ease with which they can hire these workers from unemployment. As we argued, within a firm productivity type, growing firms prefer using FT contracts: as their marginal product of labor declines toward the optimal size, worker turnover costs, which are linear in firm size, become less important. When FT contracts become shorter, this trade-off is tilted against these incentives, and firms develop a stronger preference for OE employment.

This intuition explains the policy effects on aggregate worker flows. On the one hand, workers that are currently employed under a FT contract see their position expire at a higher frequency, so overall EU rates increase, particularly driven by the FT margin (Panels (k), (l) and (m)). Moreover, as now firms increasingly prefer to hire through the OE margin, job-filling rates decline in the FT market (Panel (i)). Even though UE rates in the OE market increase slightly (Panel (j)), due to the moderate degree of substitutability between skill levels, and that promotion rates increase sharply (Panel (n)), due to firms' increased preference toward longer-tenure workers, these forces are not quantitatively strong enough to overturn an overall decline in UE rates (Panel (h)). The combination of declining job-filling rates and increasing job-destruction rates lead to the sharp

Figure 3: Effects of FT contract duration on selected equilibrium variables.



Notes: For all panels, the horizontal axis represents $1/s_{FT}^{W}$, and is measured in quarters. The plots shows different stationary solutions of the model, keeping all parameters fixed at their baseline calibration values except for s_{FT}^{W}. The red dashed vertical line shows the expected duration in the baseline calibration. Both the "aggregate output" and the "output per worker" panels are normalized to one for the baseline calibration. For the computation of EU and UE rates, see Appendix A.4.

increase in unemployment.[46] Finally, as firms now prefer to employ more workers under an OE contract, and because it is only OE workers the ones that can accumulate human capital on the job, the share of high-skill workers in the economy increases substantially, from 44.6% in the baseline to 50.2% under the policy (Panel (g)). This increase is particularly pronounced among low-type (φ_2) firms. It is among these firms, too, that the decline in the share of temporary employment is most pronounced (their average temporary share gets reduced from nearly 25% to less than 10%).

Table 6: Policy Results

	(A)	(B)	(C)
	Short duration (1 month)	*Baseline* (5 months)	*Long duration* (1 year)
Measure of incumbent firms	0.087	1.000	0.118
... Share of type-φ_1 firms	14.4%	12.9%	12.5%
Average firm size	6.7	7.1	7.2
... Relative size φ_2 firms	0.57	0.61	0.63
Firm entry rate (annualized)	8.9%	8.6%	8.6%
Average temporary share	8.2%	24.6%	32.0%
... within φ_1 firms	2.8%	4.6%	4.5%
... within φ_2 firms	9.8%	27.6%	38.2%
Share of H workers	49.9%	44.6%	42.7%
... within φ_1 firms	64.4%	63.5%	63.3%
... within φ_2 firms	45.5%	40.0%	38.1%
UE rate (total)	11.1%	17.6%	18.5%
... UE rate (FT)	10.0%	16.5%	17.2%
... UE rate (OE)	1.1%	1.1%	1.3%
EU rate (total)	8.0%	5.2%	3.4%
... EU rate (FT)	74.2%	15.3%	6.9%
... EU rate (OE)	2.1%	1.9%	1.8%
Promotion rate	14.5%	4.6%	3.0%
Unemployment rate	41.8%	22.5%	15.1%
Aggregate output (baseline=1)	0.767	1.000	1.089
Output loss from misallocation	7.4%	6.8%	6.6%
... of which between-firm misallocation	52.6%	53.3%	54.7%
Output per worker (baseline=1)	1.021	1.000	0.995
... firm size channel	1.017	1.000	0.995
... firm selection channel	1.021	1.000	0.994
... reallocation channel	0.982	1.000	1.005

Notes: Column (B) corresponds to the baseline calibration; in column (C), we set $s^W_{FT} = 1/4$ so that FT contracts expire on average after 1 year; in column (A) we set $s^W_{FT} = 3$, so that FT contracts expire on average after 1 month. The last three rows of the table compute output per worker when allowing only one productivity component to adjust at a time, following equation (A.5). UE, EU and promotion rates are quarterly figures, whereas the firm entry rate is an annual figure. For the computation of EU and UE rates, see Appendix A.4.

[46] The joint dynamics of the aggregate unemployment rate are written in the system of equations (A.17a)-(A.17d) in the Appendix.

5.1.2 The Measure of Firms and its Composition

As a byproduct of the changing firm incentives described above there are also important consequences for the total measure of operating firms and the distribution of firms across productivity levels. First, with the FT contracts being less useful there is a reduction in the value of firm incumbency, which reduces firm entry and the total measure of operating firms F (Panel (a)). Note that despite the decrease in the measure of firms, the employment rate declines so much that average firm size (E/F) is lower (Panel (c)). Second, under the policy, there are strong selection effects, with an increase in the turnover of firms (Panel (e)), a higher share in equilibrium of the firms of more productive permanent type (Panel (b)), and an increase in the relative size of these firms (Panel (d)). Because firms of less productive types φ_2 rely more on FT workers, the worsening of these contracts damages these firms relatively more, inducing a higher exit risk among them.

5.2 A Closer Look: Effects on Productivity, Misallocation, and Output

The various forces identified above shape the patterns of aggregate productivity, misallocation, and the output losses that we see in the three bottom panels of Figure 3 and in the bottom block of Table 6. We next explain what is behind these changes.

5.2.1 Decomposing Aggregate Productivity

To further understand the policy effects on aggregate productivity (Panel (q) in Figure 3), we decompose aggregate productivity in three elements. Let us start by defining aggregate output $\mathcal{Y}$ as:

$$\mathcal{Y} \equiv \sum_{n_H=0}^{+\infty} \sum_{n_L=0}^{+\infty} \sum_{z\in\mathbb{Z}} \sum_{\varphi\in\Phi} Y(n_H, n_L, z, \varphi)\, F\widetilde{f}(n_H, n_L, z, \varphi) \tag{33}$$

where $F > 0$ is the total measure of incumbent firms, $\widetilde{f}(n_H, n_L, z, \varphi) \in (0, 1)$ is the share of incumbents of type φ and productivity z with $n_H \equiv n_{OEH}$ high-skill workers and $n_L \equiv n_{FT} + n_{OEL}$ low-skill workers, and $Y(\cdot)$ is the production function defined in equation (2). Our object of interest is aggregate productivity, defined as output per worker, $\mathcal{Y}/E$, where $E > 0$ is the aggregate measure of employed workers, satisfying $E + U = 1$ (where U is the unemployment rate).

To decompose productivity, we need to introduce some new notation. First, let $n \equiv n_H + n_L$ be total firm employment. Second, we denote by $\widehat{n} \equiv n/(E/F)$ the total employment of a firm relative to the average firm size, so that if $\widehat{n} > 1$ (respectively, $\widehat{n} < 1$) the firm is larger (respectively, smaller) than the average firm in the economy. And third, we denote by $h \equiv n_H/n$ the skill share of the firm. Then, in Appendix A.5 we show that aggregate output per worker can be written as:

$$\frac{\mathcal{Y}}{E} = \frac{1}{\mathcal{E}} \sum_{z\in\mathbb{Z}} \sum_{\varphi\in\Phi} \mathcal{F}_{z,\varphi}\mathcal{R}_{z,\varphi} \tag{34}$$

In this equation, we can identify three key objects:

$$\underbrace{\mathcal{E} \equiv \left(\frac{E}{F}\right)^{1-\nu}}_{\text{(1) Firm size term}}, \qquad \underbrace{\mathcal{F}_{z,\varphi} \equiv \frac{F_{z,\varphi}}{F}}_{\text{(2) Firm selection term}}, \qquad \mathcal{R}_{z,\varphi} \equiv \underbrace{\sum_{\widehat{n}}\sum_{h}\mathbf{Y}(\widehat{n},h,z,\varphi)g_{z,\varphi}(\widehat{n},h)}_{\text{(3) Reallocation term}}$$

where $F_{z,\varphi} \geq 0$ is the measure of incumbent firms of productivity type (z,φ), so that $\sum_z\sum_\varphi F_{z,\varphi} = F$; $\mathbf{Y}(\widehat{n},h,z,\varphi) \equiv \widehat{n}^{\nu}Y(h,1-h,z,\varphi)$ with $Y(\cdot)$ given by equation (2); and $g_{z,\varphi}(\widehat{n},h) \in (0,1)$, which we define precisely in equation (A.20) of the Appendix, is the fraction of firms of productivity type (z,φ) that have relative employment $\widehat{n}$ and skill share h, so that $\sum_{\widehat{n}}\sum_h g_{z,\varphi}(\widehat{n},h) = 1$, $\forall(z,\varphi) \in \mathbb{Z}\times\Phi$.

Equation (34) shows that we can decompose aggregate productivity changes between changes in three distinct components. The first component, $1/\mathcal{E}$, reflects a *firm size channel*. This term captures productivity gains due to increasing the number of firms per worker. The increase in the number of firms per worker increases aggregate productivity due to the existence of decreasing returns to scale ($\nu < 1$). Thus, when firms are on average smaller ($\mathcal{E}$ decreases), aggregate productivity will rise, holding everything else fixed.

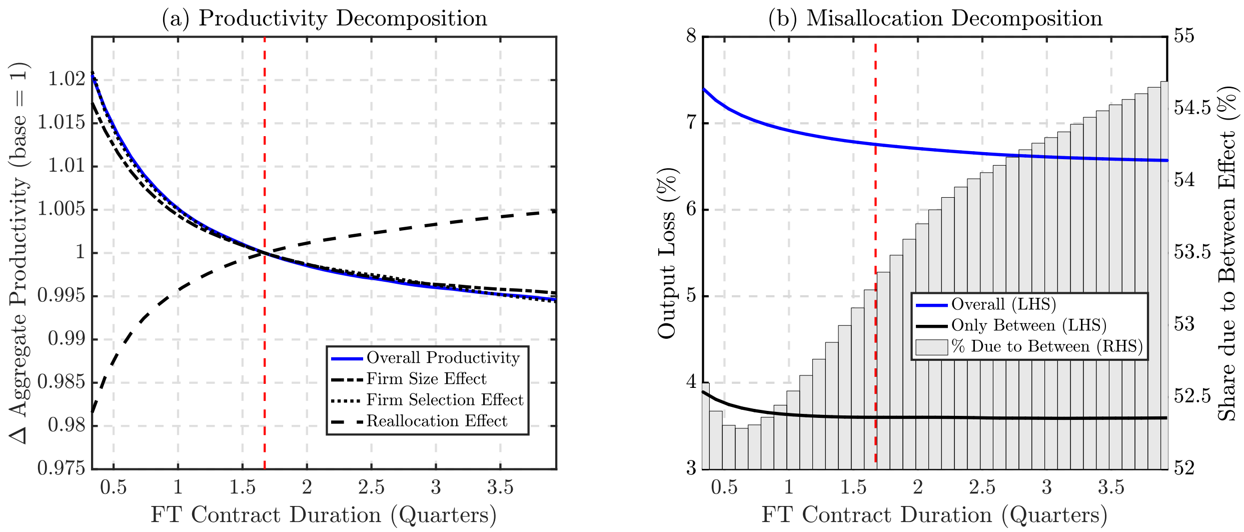
The second component of productivity, $\mathcal{F}_{z,\varphi}$, is due to a *firm selection channel*. Changes in the productivity composition of firms lead to productivity gains if those firms that are more productive become more abundant in the economy. This channel works through firm entry and exit: whenever policy prompts old unproductive firms to be replaced by new productive ones, and to the extent that the policy allows productive firms to survive more easily, aggregate productivity will rise, other things equal.

Finally, the third component reflects a *reallocation channel*. Through this channel, aggregate productivity improves in response to policy when the relative allocation of workers across firms improves, either due to the allocation of total employment (relative to average firm size) for each given share of high-skill workers, or vice versa. The joint probability $g_{z,\varphi}(\widehat{n},h)$ measures how firms are distributed in the space of relative size and human capital. In turn, we can further decompose the fraction of workers in each firm by $g_{z,\varphi}(\widehat{n},h) = g^A_{z,\varphi}(h|\widehat{n})g^B_{z,\varphi}(\widehat{n})$. The first term, $g^A_{z,\varphi}(h|\widehat{n})$, reflects a *within-firm* reallocation component, as it captures how the skill composition h changes within firms of the same productivity (z,φ) and relative size $\widehat{n}$. The second term, $g^B_{z,\varphi}(\widehat{n})$, reflects a *between-firm* reallocation component, as it captures how the relative number of workers $\widehat{n}$ changes across firms of different productivities.

With this in mind, Figure 4 shows the results of our main policy experiment by expanding on Panel (q) of Figure 3 with the productivity decomposition shown in equation (34). In particular, each line on the left-hand panel shows how aggregate productivity would have changed if only one of the three terms identified above had adjusted, but the other two had remained fixed at their baseline calibration level.[47] Our results show both the firm size and the firm selection effects

[47] In particular, the "firm size effect" line is computed as the ratio of $(1/\mathcal{E}^{CF})\sum_z\sum_\varphi \mathcal{F}^B_{z,\varphi}\mathcal{R}^B_{z,\varphi}$ to $(\mathcal{Y}/E)^B \equiv (1/\mathcal{E}^B)\sum_z\sum_\varphi \mathcal{F}^B_{z,\varphi}\mathcal{R}^B_{z,\varphi}$; the "firm selection effect" line is computed as the ratio of $(1/\mathcal{E}^B)\sum_z\sum_\varphi \mathcal{F}^{CF}_{z,\varphi}\mathcal{R}^B_{z,\varphi}$ to $(\mathcal{Y}/E)^B$; and the "reallocation effect" line is computed as the ratio of $(1/\mathcal{E}^B)\sum_z\sum_\varphi \mathcal{F}^B_{z,\varphi}\mathcal{R}^{CF}_{z,\varphi}$ to $(\mathcal{Y}/E)^B$, where the superscripts B and CF denote "baseline" and "counterfactual", respectively. Therefore, visually, these three lines need not average out to the "overall productivity" line, which is computed as the ratio of $(\mathcal{Y}/E)^{CF} \equiv (1/\mathcal{E}^{CF})\sum_z\sum_\varphi \mathcal{F}^{CF}_{z,\varphi}\mathcal{R}^{CF}_{z,\varphi}$ to $(\mathcal{Y}/E)^B$.

Figure 4: Decomposition of the effects of FT duration on productivity.



Notes: The horizontal axis represents $1/s_{FT}^{W}$, and is measured in quarters. The plots shows different stationary solutions of the model, keeping all parameters fixed at their baseline calibration values except for s_{FT}^{W}. The dashed vertical line shows the expected duration in the baseline calibration. Panel (a) shows the policy results when, following the decomposition in equation (34), we adjust only one of three terms at a time (the one visible in the legend), and keep the other two terms fixed at their baseline calibration. Each line is normalized to one at the baseline duration. Panel (b) shows the total output losses, in levels, due to misallocation (solid line) and output losses only due to between-firm misallocation (dashed line), measured on the left-hand side, together with the share of the overall losses that are accounted for by the between component (bars), measured on the right-hand side.

contribute positively to productivity as the average duration of FT contracts decline, while the reallocation effects contributes negatively. Specifically, as FT contract duration decreases, firms become on average smaller (i.e. $\mathcal{E}$ decreases) and the new distribution of firms features a higher share of productive firms (i.e. $\mathcal{F}_{z,\varphi}$ increases for higher (z, φ) states), both of which help in making the economy more productive on the aggregate. At the same time, however, the allocation of employment across firms worsens.

5.2.2 Changes in Misallocation

Next, we ask: to which extent does the worsening in the allocation of employment as a result of policy reflect a worsening in the allocation of the level and composition of employment across firms of the same productivity (within component) versus across firms of different productivity (between compnent)?

The right-hand panel in Figure 3 gives an idea of how much of this reallocation effect is due to each of these margins. To do so, we plot the overall output gap between the CE and the planner's solution (blue line, left-hand side axis), and the output gap between the planner's solution and the counterfactual output that isolates the between component (black line, left-hand side axis), which we defined at the end of Section 4.5.[48] Together with these lines, on the right-hand axis, we plot the

[48]Note that, because our planner takes the distribution of firms across productivities $F_{z,\varphi}$ as given but chooses the aggregate measures of workers of each skill, (E_H, E_L), the planner's output reflects a Pareto frontier that is *invariant* to policy for each given distribution of productivities. Thus, the output gaps shown in the figure reflect the CE allocation

ratio of these two lines, i.e. the contribution of between-firm misallocation to overall misallocation.

Our results reveal that, as FT contract duration decreases, all sources of misallocation worsen, but the within component becomes a larger contributor to welfare losses. Namely, a policy that makes FT contracts shorter would increase all forms of employment misallocation, but would hurt particularly strongly the allocation of workers between firms of the same type. As duration increases, however, we see that the share of output losses due to the between component stalls (the blue line becomes flat), so all increases in welfare are due to a better allocation of workers between firms of the same type. In the limit as the two contracts become ex-post identical (i.e. equalizing their duration, $s_W^{OE} = s_W^{FT}$), virtually all of the output losses would come from misallocation of employment between firms of different productivities.

5.2.3 Heterogeneous Effects Across Firms

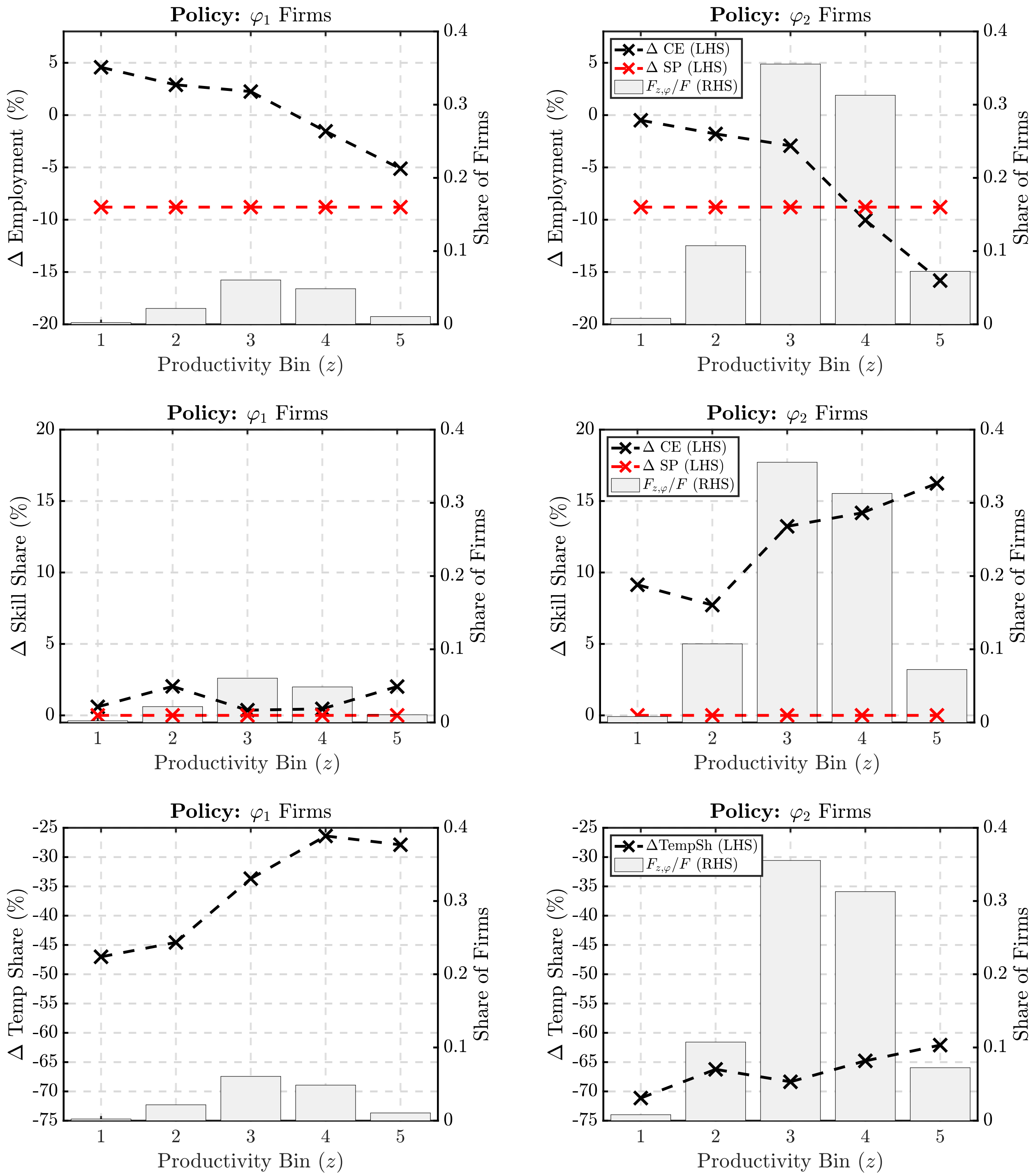
Finally, we explore the micro-level effects of the policy across different types of firms. Figure 5 shows the percentage changes in total employment, the skill share, and the temporary share for firms of different permanent types and productivities when moving from the benchmark to the policy that limits the duration of FT contracts to 1 month. When possible (i.e. for total employment and the skill share), we also show the planner's response (red crosses).

As argued above, misallocation increases as FT contract duration declines, with a relatively stronger increase in the component related to the within-firm misallocation of workers. These figures are a reflection of these results at the firm level. First, we see that all low-type (φ_2) firms reduce employment even though, among those, highly productive (z_5) firms reduce it more than they should and unproductive (z_1) do not reduce it enough. In a nutshell, the patterns of between-firm misallocation that we identified in the top panels of Figure 2 get exacerbated. Second, though all firms increase their share of skilled workers, the adjustment is much larger among low-type firms. However, as we saw in the bottom panels of Figure 2, it is precisely the high-type firms the ones that are furthest from their optimal allocation of human capital. Therefore, though beneficial for the overall levels of skill in the economy, the policy fails to sufficiently reallocate high-skill workers into the right firms. Finally, consistent with these results, all firms reduce their share of temporary employment, although the decrease is particularly pronounced among low-type firms (whose levels of temporary employment are overall much higher).

Figure 6 offers a similar description but across employment bins instead of productivity groups. The figure shows the percentage change in the average temporary share and skill share across firms of different productivities within each employment bin, for both high-type firms (blue lines) and low-type firms (red lines). Once again, we notice that the increase in the skill share is predominantly stronger among low-type firms, but particularly small ones. Large firms, by contrast, barely change their allocation of human capital. Similarly, the firms that react most in temporary employment are the small and unproductive ones, and these effects weaken as firms get larger.

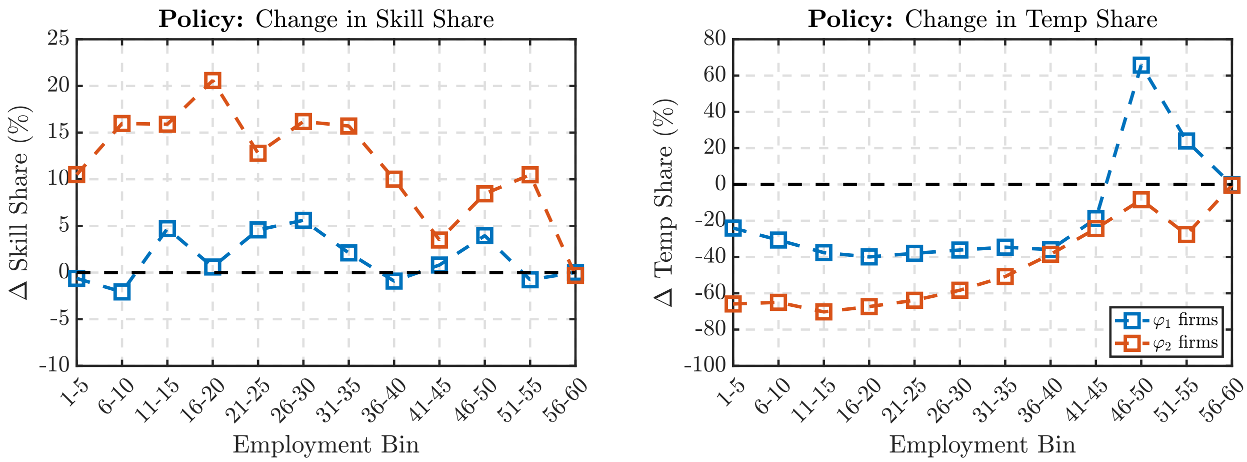
getting closer or further away from the Pareto frontier, but not movements in the frontier itself.

Figure 5: Heterogeneous Effects of the Policy by Productivity Groups.



Notes: These figures show the percentage change in employment (top row), the skill share (middle row) and the temporary share (bottom row) for high-type (left column) and low-type (right column) firms and for each transitory productivity bin (bins on the horizontal axis) when moving from the baseline duration of FT contracts to a policy that limits FT contracts to 1 month. When available (i.e. for total employment and the skill share), we also show the percentage change for the planner.

Figure 6: Heterogeneous Effects of the Policy by Employment: Temporary Share and Skill Share.



Notes: These figures show the percentage change in the average skill share (left panel) and the average temporary share (right row) across different firms that belong to the same size bin that result from moving from the baseline duration of FT contracts to a policy that limits FT contracts to 1 month.

6 Conclusion

Many labor markets are characterized by a dual structure, whereby firms are allowed to offer both open-ended (OE) contracts of long duration, and fixed-term (FT) contracts of fixed duration. Using rich administrative balance-sheet data for Spain (2004-2019), we document that there exists a high degree of heterogeneity in the usage of FT contracts between firms. We find that an overwhelming majority of the cross-sectional variation in the temporary share, i.e. in the ratio of workers employed with an FT to the total number of employees, is due to firm-level characteristics. In particular, though larger firms exhibit lower levels of temporary employment when comparing between firms within the same sector, region and time period, this relationship is reversed when looking at within-firm variation, suggesting that firms use FT contracts in their path toward their optimal size.

To rationalize these facts, and to quantify the implications of dual labor markets for aggregate productivity and the distribution of firms, we build and calibrate a firm-dynamics search-and-matching model of multi-worker firms. In the model, firms of different ex-ante permanent productivity types seek to employ workers with the prospect that they will accumulate skills on the job. To do so, firms offer dynamic long-term OE and FT contracts to their new hires, specifying trajectories for wages, layoff and promotion rates. We calibrate the model to our Spanish data and target, among other moments, the employment-to-unemployment and unemployment-to-employment worker transition rates by contract type, as well as the empirically-observed between-firm and within-firm relationships between the temporary share and firm size. The calibrated model delivers that, in order to rationalize the between- and within-firm patterns, permanently larger and more productive firms have a stronger preference for high-skill workers. As workers can only accumulate human capital on the job while employed in an open-ended position, this rationalizes that these firms rely less on temporary employment in the cross-section of firms. However, within the firm, temporary contracts are useful to grow fast and, when closer to their optimal size, firms rely more on them as the costs in worker turnover are outweighed by the benefits that these contracts offer in terms of high job-filling rates. In spite of this flexibility, however, the dual labor market structure that can be sustained thanks to search frictions is quite detrimental for output misallocation: we quantify misallocation losses to be of around 6.8% of aggregate output, half of which come from misallocation of total employment across firms of different productivities, and the other half from misallocation of human capital within the firm.

Finally, we explore the implications of dual labor markets for economic aggregates. We find that reducing the average duration of FT contracts is effective in lowering the share of temporary employment in the economy and to increase aggregate productivity through strong firm size and firm selection effects. However, this policy is ill-advised as it increases unemployment and decreases overall welfare via the misallocation of employment both within and across firms. Misallocation worsens proportionally more along the within-firm margin. A policy that would end labor market duality would be welfare improving and lead, in the limit, to an optimal allocation of human capital within the firm (although between firm misallocation of employment would remain due to the presence of search frictions).

All in all, our paper emphasizes that the firm side is an important dimension to consider in order to quantify the aggregate implications of labor market duality, a phenomenon which is pervasive in both emerging and developed economies.

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Dual Labor Markets and the Equilibrium Distribution of Firms

by Josep Pijoan-Mas and Pau Roldan-Blanco

Appendix Materials

A Derivations and Proofs

A.1 Proof of Proposition 1

Proof. Let $\overline{\mathcal{C}} = (\overline{c}_{ij})_{(i,j)\in\mathcal{I}\times\mathcal{J}}$ denote a given policy, where $\overline{c}_{ij} \equiv \left\{\overline{w}_{ij}, \overline{\delta}_{ij}, \overline{p}, \overline{W}'_{ij}(\vec{n}', z')\right\}$. Then, we can re-write the problem of a type-φ firm in (9)-(10a)-(10b) as follows:

$$\boldsymbol{J}(\vec{n}, z, \varphi, \vec{W}) = \max_{\overline{\mathcal{C}}} \; \widetilde{\boldsymbol{J}}(\vec{n}, z, \varphi, \vec{W}|\overline{\mathcal{C}}) \quad \text{subject to} \begin{cases} \boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \overline{\mathcal{C}}) \geq W_{ij}, & \forall (i,j) \\ \overline{W}'_{ij}(\vec{n}', z') \geq \boldsymbol{U}, & \forall (\vec{n}', z'), \;\; \forall (i,j) \end{cases}$$

where:

$$\begin{aligned} \widetilde{\boldsymbol{J}}(\vec{n}, z, \varphi, \vec{W}|\overline{\mathcal{C}}) \equiv \frac{1}{\overline{\rho}(\vec{n}, z)} \Bigg\{ & Y(\vec{n}, z, \varphi) - \xi n_{FT} \overline{p}^{\vartheta} + \sum_{i\in\mathcal{I}} \Bigg[\sum_{j\in\mathcal{J}} \Bigg(-\overline{w}_{ij} n_{ij} - \chi n_{ij} \overline{\delta}_{ij}^{\psi} \\ & + n_{ij} \left(\overline{\delta}_{ij} + s_i^W\right) \boldsymbol{J}\left(\vec{n}_{ij}^-, z, \varphi, \vec{W}'(\vec{n}_{ij}^-, z)\right) \Bigg) + \eta_i \left(\overline{W}'_{iL}(\vec{n}_{iL}^+, z)\right) \boldsymbol{J}\left(\vec{n}_{iL}^+, z, \varphi, \vec{W}'(\vec{n}_{iL}^+, z)\right) \Bigg] \\ & + \overline{p} n_{FT} \boldsymbol{J}\left(\vec{n}^p, z, \varphi, \vec{W}'(\vec{n}^p, z)\right) + n_{OEL} \tau \, \boldsymbol{J}\left(\vec{n}^\tau, z, \varphi, \vec{W}'(\vec{n}^\tau, z)\right) \\ & + \sum_{z'=1}^{k} \lambda(z'|z) \boldsymbol{J}\left(\vec{n}, z', \varphi, \vec{W}'(\vec{n}, z')\right) \Bigg\} \end{aligned} \tag{A.1}$$

where we have defined:

$$\overline{\rho}(\vec{n}, z) \;\equiv\; \rho + s^F + n_{FT}\overline{p} + n_{OEL}\tau + \sum_{i\in\mathcal{I}} \left[\eta_i \left(\overline{W}'_{iL}(\vec{n}_{iL}^+, z)\right) + \sum_{j\in\mathcal{J}} n_{ij} \left(\overline{\delta}_{ij} + s_i^W\right) \right]$$

as the effective discount rate of the firm. By monotonicity of preferences, the promise-keeping constraint (10a) holds with equality: $\boldsymbol{W}_{ij}(\vec{n}, z, \varphi; \overline{\mathcal{C}}) = W_{ij}$, $\forall (i,j) \in \mathcal{I}\times\mathcal{J}$. Imposing this into equation (8) allows us to solve for wages:

$$\begin{aligned} \overline{w}_{ij} = \overline{\rho}(\vec{n}, z) W_{ij} - \Bigg[& (\overline{\delta}_{ij} + s_i^W + s^F)\boldsymbol{U} + (n_{ij} - 1)\left(\overline{\delta}_{ij} + s_i^W\right) \overline{W}'_{ij}(\vec{n}_{ij}^-, z) \\ & + \sum_{(i',j')\neq(i,j)} n_{i'j'} (\overline{\delta}_{i'j'} + s_{i'}^W) \overline{W}'_{ij}(\vec{n}_{i'j'}^-, z) \end{aligned}$$

$$+ n_{FT}\overline{p}\left(\mathbf{1}[i=FT]\frac{\overline{W}'_{OEL}(\vec{n}^p,z)+(n_{FT}-1)\overline{W}'_{FT}(\vec{n}^p,z)}{n_{FT}}+\mathbf{1}[i=OE]\overline{W}'_{OE,j}(\vec{n}^p,z)\right)$$

$$+ n_{OEL}\tau\left(\mathbf{1}[(i,j)=(OE,L)]\frac{\overline{W}'_{OEH}(\vec{n}^\tau,z)+(n_{OEL}-1)\overline{W}'_{OEL}(\vec{n}^\tau,z)}{n_{OEL}}+\mathbf{1}[(i,j)\neq(OE,L)]\overline{W}'_{ij}(\vec{n}^\tau,z)\right)$$

$$+ \sum_{i'\in\mathcal{I}}\eta_{i'}\left(\overline{W}'_{i'L}(\vec{n}^+_{i'L},z)\right)\overline{W}'_{ij}(\vec{n}^+_{i'L},z)+\sum_{z'=1}^{k}\lambda(z'|z)\overline{W}'_{ij}(\vec{n},z')\Bigg] \tag{A.2}$$

where $\mathbf{1}[\cdot]$ is an indicator function. Define the joint surplus under policy $\overline{\mathcal{C}}$ as:

$$\widetilde{\boldsymbol{\Sigma}}\left(\vec{n},z,\varphi,\vec{W}\middle|\overline{\mathcal{C}}\right)\equiv\widetilde{\boldsymbol{J}}\left(\vec{n},z,\varphi,\vec{W}\middle|\overline{\mathcal{C}}\right)+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}n_{ij}W_{ij}$$

Likewise, define the *maximized* joint surplus as:

$$\boldsymbol{\Sigma}(\vec{n},z,\varphi,\vec{W})\equiv\max_{\overline{\mathcal{C}}}\left\{\widetilde{\boldsymbol{\Sigma}}\left(\vec{n},z,\varphi,\vec{W}\middle|\overline{\mathcal{C}}\right),\ \text{s.t. } \overline{W}'_{ij}(\vec{n}',z')\geq\boldsymbol{U},\ \forall(\vec{n}',z'),\ \forall(i,j)\right\}$$

Plugging (A.2) inside (A.1) and collecting terms:

$$\underbrace{\widetilde{\boldsymbol{J}}(\vec{n},z,\varphi,\vec{W}|\overline{\mathcal{C}})+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}n_{ij}W_{ij}}_{=\widetilde{\boldsymbol{\Sigma}}\left(\vec{n},z,\varphi,\vec{W}|\overline{\mathcal{C}}\right)}=\frac{1}{\overline{\rho}(\vec{n},z)}\Bigg\{Y(\vec{n},z,\varphi)-\xi n_{FT}\overline{p}^{\vartheta}+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\left(n_{ij}(\overline{\delta}_{ij}+s^W_i+s^F)\boldsymbol{U}-\chi n_{ij}\overline{\delta}^{\psi}_{ij}\right)$$

$$+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}n_{ij}(\overline{\delta}_{ij}+s^W_i)\underbrace{\left(\boldsymbol{J}(\vec{n}^-_{ij},z,\varphi,\vec{W}'(\vec{n}^-_{ij},z))+(n_{ij}-1)\overline{W}'_{ij}(\vec{n}^-_{ij},z)+\sum_{(i',j')\neq(i,j)}n_{i'j'}\overline{W}'_{i'j'}(\vec{n}^-_{i'j'},z)\right)}_{=\boldsymbol{\Sigma}\left(\vec{n}^-_{ij},z,\varphi,\vec{W}'(\vec{n}^-_{ij},z)\right)}$$

$$+\underbrace{\sum_{i\in\mathcal{I}}\left[\sum_{j\in\mathcal{J}}n_{ij}\sum_{i'\in\mathcal{I}}\eta_{i'}\left(\overline{W}'_{i'L}(\vec{n}^+_{i'L},z)\right)\overline{W}'_{ij}(\vec{n}^+_{i'L},z)+\eta_i\left(\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right)\boldsymbol{J}(\vec{n}^+_{iL},z,\varphi,\vec{W}'(\vec{n}^+_{iL},z))\right]}_{[*]}$$

$$+\ n_{FT}\overline{p}\underbrace{\left(\boldsymbol{J}(\vec{n}^p,z,\varphi,\vec{W}'(\vec{n}^p,z))+(n_{FT}-1)\overline{W}'_{FT}(\vec{n}^p,z)+(n_{OEL}+1)\overline{W}'_{OEL}(\vec{n}^p,z)\right)}_{=\boldsymbol{\Sigma}\left(\vec{n}^p,z,\varphi,\vec{W}'(\vec{n}^p,z)\right)}$$

$$+\ n_{OEL}\tau\underbrace{\left(\boldsymbol{J}(\vec{n}^\tau,z,\varphi,\vec{W}'(\vec{n}^\tau,z))+(n_{OEL}-1)\overline{W}'_{OEL}(\vec{n}^\tau,z)+(n_{OEH}+1)\overline{W}'_{OEH}(\vec{n}^\tau,z)\right)}_{=\boldsymbol{\Sigma}\left(\vec{n}^\tau,z,\varphi,\vec{W}'(\vec{n}^\tau,z)\right)}$$

$$+\sum_{z'=1}^{k}\lambda(z'|z)\underbrace{\left(\boldsymbol{J}(\vec{n},z',\varphi,\vec{W}'(\vec{n},z'))+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}n_{ij}\overline{W}'_{ij}(\vec{n},z')\right)}_{=\boldsymbol{\Sigma}\left(\vec{n},z',\varphi,\vec{W}'(\vec{n},z')\right)}\Bigg\} \tag{A.3}$$

The term labeled [$*$] can be written as follows:

$$
\begin{aligned}
[*] &= \sum_{i\in\mathcal{I}}\left[\sum_{j\in\mathcal{J}} n_{ij}\sum_{i'\in\mathcal{I}}\eta_{i'}\left(\overline{W}'_{i'L}(\vec{n}^+_{i'L},z)\right)\overline{W}'_{ij}(\vec{n}^+_{i'L},z)+\eta_i\left(\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right)\boldsymbol{J}(\vec{n}^+_{iL},z,\varphi,\vec{W}'(\vec{n}^+_{iL},z))\right]\\
&= \sum_{i\in\mathcal{I}}\eta_i\left(\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right)\left[\boldsymbol{J}(\vec{n}^+_{iL},z,\varphi,\vec{W}'(\vec{n}^+_{iL},z))+n_{iL}\overline{W}'_{iL}(\vec{n}^+_{iL},z)+\sum_{(i',j')\neq(i,L)} n_{i'j'}\overline{W}'_{i'j'}(\vec{n}^+_{iL},z)\right]\\
&= \sum_{i\in\mathcal{I}}\eta_i\left(\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right)\left[\boldsymbol{J}(\vec{n}^+_{iL},z,\varphi,\vec{W}'(\vec{n}^+_{iL},z))+(n_{iL}+1)\overline{W}'_{iL}(\vec{n}^+_{iL},z)+\sum_{(i',j')\neq(i,L)} n_{i'j'}\overline{W}'_{i'j'}(\vec{n}^+_{iL},z)-\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right]\\
&= \sum_{i\in\mathcal{I}}\eta_i\left(\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right)\boldsymbol{\Sigma}(\vec{n}^+_{iL},z,\varphi,\vec{W}'(\vec{n}^+_{iL},z))-\sum_{i\in\mathcal{I}}\eta_i\left(\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right)\overline{W}'_{iL}(\vec{n}^+_{iL},z)
\end{aligned}
$$

Putting this back into (A.3), we find:

$$
\begin{aligned}
\widetilde{\boldsymbol{\Sigma}}\left(\vec{n},z,\varphi,\vec{W}\middle|\overline{\mathcal{C}}\right) \;=\; \frac{1}{\overline{\rho}(\vec{n},z)}\Bigg\{& Y(\vec{n},z,\varphi)-\xi n_{FT}\overline{p}^{\vartheta}+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\Bigg(n_{ij}(\overline{\delta}_{ij}+s^W_i+s^F)\boldsymbol{U}-\chi n_{ij}\overline{\delta}^{\psi}_{ij}\\
&-\eta_i\left(\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right)\overline{W}'_{iL}(\vec{n}^+_{iL},z)+n_{ij}(\overline{\delta}_{ij}+s^W_i)\boldsymbol{\Sigma}(\vec{n}^-_{ij},z,\varphi,\vec{W}'(\vec{n}^-_{ij},z))\\
&+\eta_i\left(\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right)\boldsymbol{\Sigma}(\vec{n}^+_{iL},z,\varphi,\vec{W}'(\vec{n}^+_{iL},z))\Bigg)+n_{FT}\overline{p}\boldsymbol{\Sigma}(\vec{n}^p,z,\varphi,\vec{W}'(\vec{n}^p,z))\\
&+n_{OEL}\tau\boldsymbol{\Sigma}(\vec{n}^\tau,z,\varphi,\vec{W}'(\vec{n}^\tau,z))+\sum_{z'=1}^{k}\lambda(z'|z)\boldsymbol{\Sigma}(\vec{n},z',\varphi,\vec{W}'(\vec{n},z'))\Bigg\}
\end{aligned}
\tag{A.4}
$$

Note that the right-hand side of (A.4) is independent of $\vec{W}$ and w, so we can omit this dependence from $\boldsymbol{\Sigma}$ in equation (A.4), and further simplify the equation into:

$$
\begin{aligned}
\widetilde{\boldsymbol{\Sigma}}\left(\vec{n},z,\varphi\middle|\overline{\mathcal{C}}_\Sigma\right) \;=\; &\frac{1}{\overline{\rho}(\vec{n},z)}\Bigg\{Y(\vec{n},z,\varphi)-\xi n_{FT}\overline{p}^{\vartheta}+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\Bigg(n_{ij}(\overline{\delta}_{ij}+s^W_i+s^F)\boldsymbol{U}-\chi n_{ij}\overline{\delta}^{\psi}_{ij}\\
&-\eta_i\left(\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right)\overline{W}'_{iL}(\vec{n}^+_{iL},z)+n_{ij}(\overline{\delta}_{ij}+s^W_i)\boldsymbol{\Sigma}(\vec{n}^-_{ij},z,\varphi)+\eta_i\left(\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right)\boldsymbol{\Sigma}(\vec{n}^+_{iL},z,\varphi)\Bigg)\\
&+n_{FT}\overline{p}\boldsymbol{\Sigma}(\vec{n}^p,z,\varphi)+n_{OEL}\tau\boldsymbol{\Sigma}(\vec{n}^\tau,z,\varphi)+\sum_{z'=1}^{k}\lambda(z'|z)\boldsymbol{\Sigma}(\vec{n},z',\varphi)\Bigg\}
\end{aligned}
\tag{A.5}
$$

Thus, out of the full set $\overline{\mathcal{C}}=\left\{\overline{w}_{ij},\overline{\delta}_{ij},\overline{p},\overline{W}'_{ij}(\vec{n}',z')\right\}_{(i,j)\in\mathcal{I}\times\mathcal{J}}$, only the elements

$$
\overline{\mathcal{C}}_\Sigma\equiv\left\{\overline{\delta}_{ij},\overline{p},\overline{W}'_{iL}(\vec{n}^+_{iL},z)\right\}_{(i,j)\in\mathcal{I}\times\mathcal{J}}\subset\overline{\mathcal{C}}
$$

matter for the joint surplus. The optimal contract is then:

$$\overline{\mathcal{C}}_{\Sigma}^{*} = \arg\max_{\overline{\mathcal{C}}_{\Sigma}} \left\{ \widetilde{\boldsymbol{\Sigma}}\left(\vec{n}, z, \varphi \middle| \overline{\mathcal{C}}_{\Sigma}\right) \text{ s.t. } W'_{iL}(\vec{n}^{+}_{iL}, z) \geq \boldsymbol{U},\ \forall i \in \mathcal{I} \right\} \tag{A.6}$$

Wages $\{\overline{w}_{ij}\}_{(i,j)\in\mathcal{I}\times\mathcal{J}}$ are given by equation (A.2), while the remaining continuation values are chosen to be consistent with the solution to problem (A.6), as explained in the main text. Summing up: by expressing the firm's problem in terms of continuation promises, we have shown that the optimal contract must maximize the joint surplus. Conversely, for any combination of continuation promises that maximizes the joint surplus, there is a unique wage and set of outstanding promises that maximizes the firm's value subject to the promise-keeping constraint. ■

A.2 Distribution Dynamics

A.2.1 Definitions

Let $f_t(\vec{n}, z, \varphi) > 0$ be the measure of type-φ firms with employment vector $\vec{n} = \{n_{ij}\} \in \mathbb{N}$ and idiosyncratic productivity $z \in \mathbb{Z}$, at time t. These firms seek to hire new workers of type (i, L) in market segment $W'_{iL}(\vec{n}^{+}_{iL}, z, \varphi)$ for each contract type $i \in \mathcal{I}$. We denote the total measure of active firms by $F_t \equiv \sum_{\vec{n}} \sum_{z} \sum_{\varphi} f_t(\vec{n}, z, \varphi)$, and the measure of potential entrants by $F^{e}_{t} > 0$.[49] Both of these aggregate measures are endogenous objects, and are determined in equilibrium to be consistent with the sorting patterns of firms and workers. Let $\theta_i(\vec{n}, z, \varphi)$ denote the equilibrium tightness in the market of firms of type $(\vec{n}, z, \varphi)$ looking to hire an additional (low-skill) worker under contract i, with

$$\theta_i(\vec{n}, z, \varphi) = \mu_i^{-1}\left(\frac{\rho \boldsymbol{U} - b}{W'_{iL}(\vec{n}^{+}_{iL}, z, \varphi) - \boldsymbol{U}}\right) \tag{A.7}$$

by equation (7). Market tightness must adjust to guarantee that:

$$u_{it}(\vec{n}^{+}_{iL}, z, \varphi)\theta_i(\vec{n}, z, \varphi) = f_t(\vec{n}, z, \varphi) \tag{A.8}$$

for all t, where $u_{it}(\vec{n}^{+}_{iL}, z, \varphi)$ is the measure of unemployed workers looking to be hired in a type-i contract by a firm in state $(\vec{n}, z, \varphi)$. In words, this condition guarantees that, for every state and contract type, the total number of jobs created by firms in that state equals the total number of jobs found by workers in the corresponding submarket.[50]

Let $e_{ij,t}(\vec{n}, z, \varphi)$ be the measure of workers of type (i, j) employed by firms of type $(\vec{n}, z, \varphi)$ at time t. The assignment of workers to firms satisfies:

[49] The symbol $\sum_{\vec{n}}$ expresses an element-wise sum, i.e. summing over all n_{OEH}, n_{OEL} and n_{FT}.

[50] To see this, multiply both sides of equation (A.8) by $\eta_i(\theta_i(\vec{n}, z, \varphi))$ to write:

$$\underbrace{u_{it}(\vec{n}^{+}_{iL}, z, \varphi)\mu_i(\theta_i(\vec{n}, z, \varphi))}_{\text{Jobs found by workers}} = \underbrace{f_t(\vec{n}, z, \varphi)\eta_i(\theta_i(\vec{n}, z, \varphi))}_{\text{Jobs created by firms}}$$

where we have used the identify $\eta_i(\theta)\theta = \mu_i(\theta)$.

$$e_{ij,t}(\vec{n},z,\varphi) = n_{ij} f_t(\vec{n},z,\varphi) \tag{A.9}$$

by construction. The unit measure of workers must be either matched with a firm or unmatched and searching. This gives us a formula for the unemployment rate, $U_t = 1 - E_t$, where:

$$E_t = \sum_i \sum_j \sum_z \sum_\varphi n_{ij} f_t(\{n_{ij}\},z,\varphi).$$

A.2.2 Kolmogorov Forward Equations

Next, we write down the dynamics of the distribution of firms. The rate of change in the measure of firms in state $(\vec{n},z,\varphi)$ is:

$$\begin{aligned}\frac{\partial f_t(\vec{n},z,\varphi)}{\partial t} &= \sum_{i\in\mathcal{I}} \eta_i\big(W'_{iL}(\vec{n}^-_{iL},z,\varphi)\big) f_t(\vec{n}^-_{iL},z,\varphi) + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}(n_{ij}+1)\Big(\delta_{ij}(\vec{n}^+_{ij},z,\varphi)+s^W_i\Big) f_t(\vec{n}^+_{ij},z,\varphi)\\ &+ (n_{FT}+1)p(\vec{n}^-_p,z,\varphi)f_t(\vec{n}^-_p,z,\varphi) + (n_{OEL}+1)\tau f_t(\vec{n}^-_\tau,z,\varphi) + \sum_{\widehat{z}\neq z}\lambda(z|\widehat{z})f_t(\vec{n},\widehat{z},\varphi)\\ &- \Big[s^F + \sum_{i\in\mathcal{I}}\eta_i\big(W'_{iL}(\vec{n},z,\varphi)\big) + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} n_{ij}\Big(\delta_{ij}(\vec{n},z,\varphi)+s^W_i\Big) + n_{FT}p(\vec{n},z,\varphi) + n_{OEL}\tau + \sum_{\widehat{z}\neq z}\lambda(\widehat{z}|z)\Big] f_t(\vec{n},z,\varphi).\end{aligned} \tag{A.10}$$

The first two rows on the right-hand side of equation (A.10) give inflows into state $(\vec{n},z,\varphi)$. Inflows come from firms with $\vec{n}^-_{iL} \equiv (n_{iL}-1,n_H)$ that hire a worker with contract i, firms with $\vec{n}^+_{ij} \equiv (n_{ij}+1,\vec{n}_{-(ij)})$ that fire a type-(i,j) worker, firms with $\vec{n}^-_p \equiv (n_{OEH},n_{OEL}-1,n_{FT}+1)$ that promote an FT worker into an OE contract, firms with $\vec{n}^-_\tau \equiv (n_{OEH}-1,n_{OEL}+1,n_{FT})$ for whom a low-skill OE worker has a skill upgrade, and firms at $\vec{n} = (n_{ij},\vec{n}_{-(ij)})$ that transition into productivity z from some $\widehat{z}\neq z$. The last row of equation (A.10) gives the corresponding outflows.[51] On the other hand, the dynamics of inactive firms are:

$$\begin{aligned}\frac{\partial F^e_t}{\partial t} &= s^F F_t + \sum_{\varphi\in\Phi}\sum_{z\in\mathbb{Z}}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\Big(\delta_{ij}(\vec{n}^e_{ij},z,\varphi)+s^W_i\Big) f_t(\vec{n}^e_{ij},z,\varphi)\\ &\quad - F^e_t \sum_{\varphi\in\Phi}\sum_{z^e\in\mathbb{Z}}\pi_\varphi(\varphi)\pi_z(z^e)\sum_{i\in\mathcal{I}}\eta_i\big(W'_{iL}(\vec{n}^e_{iL},z^e,\varphi)\big)\end{aligned} \tag{A.11}$$

where the first term are inflows from exiting firms, the second term are inflows from firms with $\vec{n}^e_{ij} \equiv (n_{ij},\vec{n}_{-(ij)}) = (1,\vec{0})$ that lose their last remaining worker, and the third term are outflows from successful entrants.

[51]Inflows from hiring must be multiplied by $\pi_z(z)\pi_\varphi(\varphi)$ whenever they come from successful entrants, i.e. for $(n_{iL},n_H) = (1,0)$.

A.2.3 Finding the Stationary Distribution

To find the stationary distribution, we impose steady state into equations (A.10) and (A.11), i.e. $\frac{\partial f_t(\vec{n},z,\varphi)}{\partial t} = \frac{\partial F_t^e}{\partial t} = 0$. In the numerical implementation, we solve for the *share*, not the measure, of firms (active or inactive) in each state $s \equiv (n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)$. Denote the state space by $\mathbb{S}$, which takes values in $\overline{\mathbb{N}}_{OEH} \times \overline{\mathbb{N}}_{OEL} \times \overline{\mathbb{N}}_{FT} \times \{z_1, \dots, z_k\} \times \{\varphi_1, \varphi_2\}$, where we denote $\overline{\mathbb{N}}_{ij} \equiv \{0, 1, \dots, N_{ij}\}$, for some positive integer N_{ij} and each $(i,j) \in \mathcal{I} \times \mathcal{J}$. Specifically, there is one inactive state, $s = 0$ (namely, $n_{OEH} = n_{OEL} = n_{FT} = 0$), and $S \equiv \left[(N_{OEH}+1)\cdot(N_{OEL}+1)\cdot(N_{FT}+1) - 1\right] \cdot k \cdot 2$ active states.[52] Denote by $\phi(s) \in [0,1]$ the share of firms in state $s = 1, \dots, S$, such that $\sum_{s=1}^{S} \phi(s) = 1 - \phi_0$, where $\phi_0 > 0$ is the share of inactive firms. Stacking all of these shares into the column vector $\vec{\phi} \equiv [\phi_0, \phi(1), \dots, \phi(S)]^\top$, we have a system of $S+1$ linear equations, which in matrix form reads:

$$\boldsymbol{G}^\top \vec{\phi} = \vec{0} \tag{A.12}$$

The object $\boldsymbol{G}$ is a $(S+1)$-dimensional infinitesimal generator matrix, which registers inflows in the diagonal and outflows in the off-diagonal, such that:

$$\boldsymbol{G} = \begin{pmatrix} -\sum_{s\neq 0} g_{0,s} & g_{0,1} & g_{0,2} & \cdots & g_{0,S} \\ g_{1,0} & -\sum_{s\neq 1} g_{1,s} & g_{1,2} & \cdots & g_{1,S} \\ g_{2,0} & g_{2,1} & -\sum_{s\neq 2} g_{2,s} & \cdots & g_{2,S} \\ \vdots & \vdots & \ddots & \vdots & \\ g_{S,0} & g_{S,1} & g_{S,2} & \cdots & -\sum_{i\neq S} g_{S,i} \end{pmatrix}$$

The transition rates $g_{i,j}$ are built using the optimal policies, following the laws of motion stated in equations (A.10) and (A.11). To solve for $\vec{\phi}$, we write system (A.12) as $(\boldsymbol{G}^\top + \boldsymbol{I})\vec{\phi} = \vec{\phi}$, where $\boldsymbol{I}$ is a $(S+1)$-dimensional identity matrix. This means that $\vec{\phi}$ can be computed as the eigenvector of $\boldsymbol{G}^\top + \boldsymbol{I}$ that is associated with the unit eigenvalue. We exploit this fact to find our invariant distribution.

In our calibration, we have $S = 79{,}350$, so $\boldsymbol{G}$ and $\boldsymbol{I}$ are very large matrices. However, as the vast majority of transitions are illegal, $\boldsymbol{G}$ has many zero entries, so in practice we use sparsity methods to save on computing time and memory, defining $(\boldsymbol{G}, \boldsymbol{I})$ as sparse matrices. These methods cut down computing time to about 2 minutes per model solution.

A.3 Aggregate Measures of Agents

Having found the invariant distribution, we then make the following normalization:

$$\tilde{f}(s) \equiv \frac{\phi(s)}{1 - \phi_0}$$

[52] For each (z, φ) firm type, there are $(N_{OEH}+1)\cdot(N_{OEL}+1)\cdot(N_{FT}+1) - 1$ possible states for an active firm, where the "–1" is because $N_{OEH} = N_{OEL} = N_{FT} = 0$ corresponds to the inactive state.

for each state s corresponding to a point $(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)$ in the state space. In words, $\widetilde{f}(s)$ is the share of active firms in state s relative to all active firms, i.e. $\widetilde{f}(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi) \equiv f(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)/F$. To proceed, we use identity (A.9) to compute:

$$\widetilde{e}_{ij}(\vec{n}, z, \varphi) = n_{ij}\widetilde{f}(\vec{n}, z, \varphi)$$

That is, $\widetilde{e}_{ij} \equiv e_{ij}/F$ is the measure of workers of type (i, j) employed in firms of type $(\vec{n}, z, \varphi)$, as a share of the total measure of active firms. From this we can find $\widetilde{E}_{ij} \equiv E_{ij}/F = \sum_{\vec{n}} \sum_z \sum_\varphi \widetilde{e}_{ij}(\vec{n}, z, \varphi)$, i.e. the total measure of employed individuals of type (i, j) per firm, and $\widetilde{E} \equiv E/F = \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \widetilde{E}_{ij}$, i.e. the average firm size.

Next, using (A.8), we know:

$$\begin{aligned}
u_{OE}(n_{OEH}, n_{OEL}+1, n_{FT}, z, \varphi) &= F\,\frac{\widetilde{f}(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)}{\theta_{OE}(n_{OEH}, n_{OEL}+1, n_{FT}, z, \varphi)} \\
u_{FT}(n_{OEH}, n_{OEL}, n_{FT}+1, z, \varphi) &= F\,\frac{\widetilde{f}(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)}{\theta_{FT}(n_{OEH}, n_{OEL}, n_{FT}+1, z, \varphi)}
\end{aligned}$$

To make progress, in the first equation we add up both sides across all $(n_{OEH}, n_{FT}, z, \varphi)$, as well as over $n_{OEL} = 1, \ldots, N_{OE}$ (i.e. omitting $n_{OEL} = 0$). Similarly, in the second equation we add across all $(n_{OEH}, n_{OEL}, z, \varphi)$, as well as over $n_{FT} = 1, \ldots, N_{FT}$ (i.e. omitting $n_{FT} = 0$). That is:

$$\sum_{n_{OEH}} \sum_{n_{OEL}\neq 0} \sum_{n_{FT}} \sum_z \sum_\varphi u_{OE}(n_{OEH}, n_{OEL}+1, n_{FT}, z, \varphi) = F\left(\sum_{n_{OEH}} \sum_{n_{OEL}\neq 0} \sum_{n_{FT}} \sum_z \sum_\varphi \frac{\widetilde{f}(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)}{\theta_{OE}(n_{OEH}, n_{OEL}+1, n_{FT}, z, \varphi)}\right) \tag{A.13}$$

$$\sum_{n_{OEH}} \sum_{n_{OEL}} \sum_{n_{FT}\neq 0} \sum_z \sum_\varphi u_{FT}(n_{OEH}, n_{OEL}, n_{FT}+1, z, \varphi) = F\left(\sum_{n_{OEH}} \sum_{n_{OEL}} \sum_{n_{FT}\neq 0} \sum_z \sum_\varphi \frac{\widetilde{f}(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)}{\theta_{FT}(n_{OEH}, n_{OEL}, n_{FT}+1, z, \varphi)}\right) \tag{A.14}$$

Developing the left-hand side of (A.13) and (A.14) yields:

$$\begin{aligned}
\sum_{n_{OEH}} \sum_{n_{OEL}\neq 0} \sum_{n_{FT}} \sum_z \sum_\varphi u_{OE}(n_{OEH}, n_{OEL}+1, n_{FT}, z, \varphi) \;&=\; U_{OE} - \sum_{n_{OEH}} \sum_{n_{FT}} \sum_z \sum_\varphi u_{OE}(n_{OEH}, 1, n_{FT}, z, \varphi) \\
&= \underbrace{1 - E - U_{FT}}_{=U_{OE}} - \sum_z \sum_\varphi \left(\frac{F^e}{\theta_{OE}(0,1,0,z,\varphi)} + \sum_{(n_{OEH},n_{FT})\neq(0,0)} \frac{f(n_{OEH}, 0, n_{FT}, z, \varphi)}{\theta_{OE}(n_{OEH}, 1, n_{FT}, z, \varphi)}\right) \\
&= 1 - U_{FT} - F\left[\widetilde{E} + \sum_z \sum_\varphi \left(\frac{\widetilde{F}^e}{\theta_{OE}(0,1,0,z,\varphi)} + \sum_{(n_{OEH},n_{FT})\neq(0,0)} \frac{\widetilde{f}(n_{OEH}, 0, n_{FT}, z, \varphi)}{\theta_{OE}(n_{OEH}, 1, n_{FT}, z, \varphi)}\right)\right]
\end{aligned}$$

and

$$\sum_{n_{OEH}}\sum_{n_{OEL}}\sum_{n_{FT}\neq 0}\sum_{z}\sum_{\varphi} u_{FT}(n_{OEH}, n_{OEL}, n_{FT}+1, z, \varphi) = U_{FT} - \sum_{n_{OEH}}\sum_{n_{OEL}}\sum_{z}\sum_{\varphi} u_{FT}(n_{OEH}, n_{OEL}, 1, z, \varphi)$$

$$= U_{FT} - \sum_{z}\sum_{\varphi}\left(\frac{F^e}{\theta_{FT}(0,0,1,z,\varphi)} + \sum_{(n_{OEH},n_{OEL})\neq(0,0)} \frac{f(n_{OEH}, n_{OEL}, 0, z, \varphi)}{\theta_{FT}(n_{OEH}, n_{OEL}, 1, z, \varphi)}\right)$$

$$= U_{FT} - F\left[\sum_{z}\sum_{\varphi}\left(\frac{\widetilde{F}^e}{\theta_{FT}(0,0,1,z,\varphi)} + \sum_{(n_{OEH},n_{OEL})\neq(0,0)} \frac{\widetilde{f}(n_{OEH}, n_{OEL}, 0, z, \varphi)}{\theta_{FT}(n_{OEH}, n_{OEL}, 1, z, \varphi)}\right)\right]$$

respectively. Substituting these back into (A.13) and (A.14) yields:

$$1 - U_{FT} - F\left[\widetilde{E} + \sum_{z}\sum_{\varphi}\left(\frac{\widetilde{F}^e}{\theta_{OE}(0,1,0,z,\varphi)} + \sum_{(n_{OEH},n_{FT})\neq(0,0)} \frac{\widetilde{f}(n_{OEH}, 0, n_{FT}, z, \varphi)}{\theta_{OE}(n_{OEH}, 1, n_{FT}, z, \varphi)}\right)\right]$$
$$= F\left(\sum_{n_{OEH}}\sum_{n_{OEL}\neq 0}\sum_{n_{FT}}\sum_{z}\sum_{\varphi} \frac{\widetilde{f}(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)}{\theta_{OE}(n_{OEH}, n_{OEL}+1, n_{FT}, z, \varphi)}\right)$$

and

$$U_{FT} - F\left[\sum_{z}\sum_{\varphi}\left(\frac{\widetilde{F}^e}{\theta_{FT}(0,0,1,z,\varphi)} + \sum_{(n_{OEH},n_{OEL})\neq(0,0)} \frac{\widetilde{f}(n_{OEH}, n_{OEL}, 0, z, \varphi)}{\theta_{FT}(n_{OEH}, n_{OEL}, 1, z, \varphi)}\right)\right]$$
$$= F\left(\sum_{n_{OEH}}\sum_{n_{OEL}}\sum_{n_{FT}\neq 0}\sum_{z}\sum_{\varphi} \frac{\widetilde{f}(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)}{\theta_{FT}(n_{OEH}, n_{OEL}, n_{FT}+1, z, \varphi)}\right)$$

Solving for F on each equation yields:

$$F = \frac{1 - U_{FT}}{\widetilde{E} + \widetilde{U}_{OE}} \qquad \text{and} \qquad F = \frac{U_{FT}}{\widetilde{U}_{FT}} \tag{A.15}$$

respectively, where we have defined:

$$\widetilde{U}_{OE} \equiv \sum_{z}\sum_{\varphi}\left[\frac{\widetilde{F}^e}{\theta_{OE}(0,1,0,z,\varphi)} + \sum_{(n_{OEH},n_{FT})\neq(0,0)} \frac{\widetilde{f}(n_{OEH}, 0, n_{FT}, z, \varphi)}{\theta_{OE}(n_{OEH}, 1, n_{FT}, z, \varphi)} + \sum_{n_{OEH}}\sum_{n_{OEL}\neq 0}\sum_{n_{FT}} \frac{\widetilde{f}(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)}{\theta_{OE}(n_{OEH}, n_{OEL}+1, n_{FT}, z, \varphi)}\right]$$

$$\widetilde{U}_{FT} \equiv \sum_{z}\sum_{\varphi}\left[\frac{\widetilde{F}^e}{\theta_{FT}(0,0,1,z,\varphi)} + \sum_{(n_{OEH},n_{OEL})\neq(0,0)} \frac{\widetilde{f}(n_{OEH}, n_{OEL}, 0, z, \varphi)}{\theta_{FT}(n_{OEH}, n_{OEL}, 1, z, \varphi)} + \sum_{n_{OEH}}\sum_{n_{OEL}}\sum_{n_{FT}\neq 0} \frac{\widetilde{f}(n_{OEH}, n_{OEL}, n_{FT}, z, \varphi)}{\theta_{FT}(n_{OEH}, n_{OEL}, n_{FT}+1, z, \varphi)}\right]$$

Solving for U_{FT} from equation (A.15) gives $U_{FT} = \frac{\widetilde{U}_{FT}}{\widetilde{E}+\widetilde{U}_{OE}+\widetilde{U}_{FT}}$, which finally gives us the aggregate measure of active firms:

$$F = \left(\widetilde{E} + \widetilde{U}_{OE} + \widetilde{U}_{FT}\right)^{-1} \tag{A.16}$$

Equation (A.16) reflects a feasibility condition: the sum of the measures of all employed and unemployed workers must equal one, the size of the labor force. Once we have the measure of active firms F, we can compute all the remaining aggregates:

(i) the total measure of potential entrants is $F^e = \phi_0 F$;

(ii) the employment and unemployment rates are given by $E = \widetilde{E}F$ and $U = 1 - E$;

(iii) the aggregate temporary share is given by E_{FT}/E;

(iv) the share of skilled workers in the economy is E_{OEH}/E.

A.4 Worker Flow Rates

A.4.1 In the data

We compute the EU and UE quarterly rates by type of contract using data from the *Encuesta de Población Activa* (EPA), compiled by the *Instituto Nacional de Estadística* (INE), the Spanish national statistical agency. The data come at the quarterly frequency for the period 2006Q1-2019Q4. Denote by $UE^i_{t,t+1}$ the U-to-E flow from quarter t to $t+1$ into a contract of type $i = OE, FT$, and similarly for $EU^i_{t,t+1}$. E-to-E flows from an FT into an OE contract are denoted by $EE^{FtoO}_{t,t+1}$. Labor market rates are defined as follows:

$$\widehat{UE}_i^{\text{data}} \equiv \frac{\sum UE^i_{t,t+1}}{\sum U_t} \qquad \text{and} \qquad \widehat{EU}_i^{\text{data}} \equiv \frac{\sum EU^i_{t,t+1}}{\sum E^i_t}$$

where $\sum$ denotes the sum of sample weights for all observations in that category, $\sum U_t$ is the number of unemployed at time t, and $\sum E^i_t$ is the number of employed in contract type i at time t. Similarly, the promotion rate in the data is computed as follows:

$$\widehat{EE}_{FtoO}^{\text{data}} \equiv \frac{\sum EE^{FtoO}_{t,t+1}}{\sum E^{FT}_t}$$

where E^{FT}_t is the stock of FT workers in quarter t.

A.4.2 In the model

As we do not have empirical flows for within-firm skill upgrades, we categorize workers into four employment status: high-skill and low-skill employed with an OE contract, employed with an

FT contract, and unemployed. The following set of equations describes flows between these states through the lens of the model:

$$
\begin{aligned}
\frac{\partial E_{OEH}}{\partial t} &= \sum_{\vec{n}}\sum_{z}\sum_{\varphi}\left\{\tau e_{OEL}(\vec{n},z,\varphi) - \left(\delta_{OEH}(\vec{n},z,\varphi) + s^W_{OE} + s^F\right)e_{OEH}(\vec{n},z,\varphi)\right\} \\
\frac{\partial E_{OEL}}{\partial t} &= \sum_{\vec{n}}\sum_{z}\sum_{\varphi}\left\{p(\vec{n},z,\varphi)e_{FT}(\vec{n},z,\varphi) + \mu_{OE}(\vec{n}^+_{OEL},z,\varphi)u_{OE}(\vec{n}^+_{OEL},z,\varphi)\right. \\
&\qquad \left. - \left(\delta_{OEL}(\vec{n},z,\varphi) + s^W_{OE} + s^F\right)e_{OEL}(\vec{n},z,\varphi) - \tau e_{OEL}(\vec{n},z,\varphi)\right\} \\
\frac{\partial E_{FT}}{\partial t} &= \sum_{\vec{n}}\sum_{z}\sum_{\varphi}\left\{\mu_{FT}(\vec{n}^+_{FT},z,\varphi)u_{FT}(\vec{n}^+_{FT},z,\varphi)\right. \\
&\qquad \left. - \left(\delta_{FT}(\vec{n},z,\varphi) + s^W_{FT} + s^F\right)e_{FT}(\vec{n},z,\varphi) - p(\vec{n},z,\varphi)e_{FT}(\vec{n},z,\varphi)\right\} \\
\frac{\partial U}{\partial t} &= \sum_{\vec{n}}\sum_{z}\sum_{\varphi}\left\{\sum_{j=L,H}\left(\delta_{OE,j}(\vec{n},z,\varphi) + s^W_{OE} + s^F\right)e_{OE,j}(\vec{n},z,\varphi) + \left(\delta_{FT}(\vec{n},z,\varphi) + s^W_{FT} + s^F\right)e_{FT}(\vec{n},z,\varphi)\right. \\
&\qquad \left. - \sum_{i=OE,FT}\mu_i(\vec{n}^+_{iL},z,\varphi)u_i(\vec{n}^+_{iL},z,\varphi)\right\}
\end{aligned}
$$

Each of these equations collect inflows (terms with a positive sign) outflows (terms with a negative sign). A more compact way of writing the dynamical system above is:

$$
\frac{\partial E_{OEH}}{\partial t} = -\lambda_{EU_{OEH}}E_{OEH} + \lambda_{EE_{LH}}E_{OEL} \tag{A.17a}
$$

$$
\frac{\partial E_{OEL}}{\partial t} = -\left(\lambda_{EU_{OEL}} + \lambda_{EE_{LH}}\right)E_{OEL} + \lambda_{EE_{FtoO}}E_{FT} + \lambda_{UE_{OE}}U \tag{A.17b}
$$

$$
\frac{\partial E_{FT}}{\partial t} = -\left(\lambda_{EU_{FT}} + \lambda_{EE_{FtoO}}\right)E_{FT} + \lambda_{UE_{FT}}U \tag{A.17c}
$$

$$
\frac{\partial U}{\partial t} = \lambda_{EU_{OEH}}E_{OEH} + \lambda_{EU_{OEL}}E_{OEL} + \lambda_{EU_{FT}}E_{FT} - \left(\lambda_{UE_{OE}} + \lambda_{UE_{FT}}\right)U \tag{A.17d}
$$

where we have defined the following average intensities:

$$
\lambda_{EU_{OE,j}} \equiv \frac{EU_{OE,j}}{E_{OE,j}} \qquad \lambda_{EE_{FtoO}} \equiv \frac{EE_{FtoO}}{E_{FT}} \qquad \lambda_{UE_{OE}} \equiv \frac{UE_{OE}}{U}
$$

$$
\lambda_{EU_{FT}} \equiv \frac{EU_{FT}}{E_{FT}} \qquad \lambda_{UE_{FT}} \equiv \frac{UE_{FT}}{U} \qquad \lambda_{EE_{LH}} \equiv \frac{EE_{LH}}{E_{OEL}}
$$

with

$$
\begin{aligned}
EU_{OE,j} &\equiv \sum_{\vec{n}}\sum_{z}\sum_{\varphi}\left(\delta_{OE,j}(\vec{n},z,\varphi)+s^W_{OE}+s^F\right)e_{OE,j}(\vec{n},z,\varphi) & EE_{FtoO} &\equiv \sum_{\vec{n}}\sum_{z}\sum_{\varphi} p(\vec{n},z,\varphi)e_{FT}(\vec{n},z,\varphi) \\
UE_{OE} &\equiv \sum_{\vec{n}}\sum_{z}\sum_{\varphi}\mu_{OE}(\vec{n}^+_{OEL},z,\varphi)u_{OE}(\vec{n}^+_{OEL},z,\varphi) & EU_{FT} &\equiv \sum_{\vec{n}}\sum_{z}\sum_{\varphi}\left(\delta_{FT}(\vec{n},z,\varphi)+s^W_{FT}+s^F\right)e_{FT}(\vec{n},z,\varphi) \\
UE_{FT} &\equiv \sum_{\vec{n}}\sum_{z}\sum_{\varphi}\mu_{FT}(\vec{n}^+_{FT},z,\varphi)u_{FT}(\vec{n}^+_{FT},z,\varphi) & EE_{LH} &\equiv \sum_{\vec{n}}\sum_{z}\sum_{\varphi}\tau e_{OEL}(\vec{n},z,\varphi)
\end{aligned}
\tag{A.18}
$$

for $j = H, L$. Note, in particular, that since all firms face the same rate of skill upgrade, we have that $\lambda_{EE_{LH}} = \tau$. We can write system (A.17a)-(A.17d) in vector-matrix form as follows:

$$
\frac{\partial}{\partial t}\begin{bmatrix} E_{OEH} \\ E_{OEL} \\ E_{FT} \\ U \end{bmatrix} = \begin{pmatrix} -\lambda_{EU_{OEH}} & \lambda_{EE_{LH}} & 0 & 0 \\ 0 & -(\lambda_{EU_{OEL}}+\lambda_{EE_{LH}}) & \lambda_{EE_{FtoO}} & \lambda_{UE_{OE}} \\ 0 & 0 & -(\lambda_{EU_{FT}}+\lambda_{EE_{FtoO}}) & \lambda_{UE_{FT}} \\ \lambda_{EU_{OEH}} & \lambda_{EU_{OEL}} & \lambda_{EU_{FT}} & -(\lambda_{UE_{OE}}+\lambda_{EU_{FT}}) \end{pmatrix}\begin{bmatrix} E_{OEH} \\ E_{OEL} \\ E_{FT} \\ U \end{bmatrix}
$$

Setting the right-hand side to the zero vector and solving the resulting system of linear equations will give us the stationary measures in the reduced-form model. Notice that, by construction, the resulting stocks coincide exactly, by construction, with the ones derived from firm-level flows in Appendix A.3.

Using these results, we can now construct flow rates. As the model is set in continuous time, we must produce discrete-time approximations in order to have numbers that can be compared to the ones from the quarterly data. For this, we compute for each contract type $i = OE, FT$:

$$
\widehat{UE}_i^{\text{model}} = \frac{1-e^{-UE_i dt}}{U} \qquad \text{and} \qquad \widehat{EU}_i^{\text{model}} = \frac{1-e^{-\sum_{j=L,H} EU_{ij}dt}}{\sum_{j=L,H} E_{ij}}
$$

In these ratios, in the numerator we have transformed instantaneous Poisson rates into quarterly probabilities by setting $dt = 1/4$.[53] For the overall UE and EU rates, we compute:

$$
\widehat{UE}_{\text{total}}^{\text{model}} = \frac{1-e^{-\sum_i UE_i dt}}{U} \qquad \text{and} \qquad \widehat{EU}_{\text{total}}^{\text{model}} = \frac{1-e^{-\sum_i\sum_j EU_{ij}dt}}{\sum_i\sum_j E_{ij}}
$$

Similarly, to obtain the promotion rate at the quarterly frequency in the model, we compute:

$$
\widehat{EE}_{FtoO}^{\text{model}} = \frac{1-e^{-EE_{FtoO}dt}}{E_{FT}}
$$

For estimation purposes, we treat $\widehat{UE}_i^{\text{model}}$, $\widehat{EU}_i^{\text{model}}$ and $\widehat{EE}_{FtoO}^{\text{model}}$ as the direct model counterparts of $\widehat{UE}_i^{\text{data}}$, $\widehat{EU}_i^{\text{data}}$ and $\widehat{EE}_{FtoO}^{\text{data}}$, respectively.

[53] In particular, the numerator is the probability that there is *at least* one transition (i.e. one or more transitions) within a given quarter, which we compute as the complementary probability of no transitions.

A.5 Productivity Decomposition

In order to decompose aggregate productivity into the different parts shown in equation (34), we first need to transform equation (33) in two steps.

First, note that the measure $f(n_H, n_L, z, \varphi) \equiv F\widetilde{f}(n_H, n_L, z, \varphi)$ of firms in state (n_H, n_L, z, φ) can always be written as $f(n_H, n_L, z, \varphi) = F_{z,\varphi}\widetilde{f}_{z,\varphi}(n_H, n_L)$, where $\widetilde{f}_{z,\varphi}(n_H, n_L)$ denotes the share of incumbent firms of type (z, φ) with employment vector (n_H, n_L), and $F_{z,\varphi} \equiv \sum_{n_H}\sum_{n_L} f(n_H, n_L, z, \varphi)$ is the measure of firms in state (z, φ), so that $\sum_z \sum_\varphi F_{z,\varphi} = F$. Then, we can write equation (33) as:

$$\frac{\mathcal{Y}}{E} = \frac{F}{E}\left[\sum_{z\in\mathbb{Z}}\sum_{\varphi\in\Phi}\frac{F_{z,\varphi}}{F}\left(\sum_{n_H=0}^{+\infty}\sum_{n_L=0}^{+\infty} Y(n_H, n_L, z, \varphi)\widetilde{f}_{z,\varphi}(n_H, n_L)\right)\right]$$

Second, given that the production function $Y(n_H, n_L, z, \varphi)$ is homogeneous of degree ν in n_H and n_L, we can rewrite it as a function of total employment, $n \equiv n_H + n_L$, and the skill share in the firm, $h \equiv n_H/n$, given by $\mathbf{Y}(n, h, z, \varphi) \equiv n^\nu Y(h, 1-h, z, \varphi)$. Notice that, because of the discreteness of the state space, h takes values in a discrete subset of the unit interval: if $n = 1$, then $h \in \mathbb{H}_1 \equiv \{0, 1\}$; if $n = 2$, then $h \in \mathbb{H}_2 \equiv \{0, 1/2, 1\}$; if $n = 3$, then $h \in \mathbb{H}_3 \equiv \{0, 1/3, 2/3, 1\}$; and so on. Generally, denote $\mathbb{H} \equiv \bigcup_{n\in\mathbb{N}} \mathbb{H}_n \subseteq [0, 1]$.[54]

To identify the size effect, instead of firm employment n we will use firm employment relative to average firm employment, $\widehat{n} \equiv n/(E/F)$, such that

$$\mathbf{Y}(n, h, z, \varphi) = \left(\frac{E}{F}\right)^\nu \mathbf{Y}(\widehat{n}, h, z, \varphi)$$

Notice that, for given E/F, normalized firm employment $\widehat{n}$ is a random variable with discrete support, $\widehat{\mathbb{N}} \equiv \left(\frac{E}{F}\right)\mathbb{N}$. Given this, aggregate productivity $\mathcal{Y}/E$ can be written as:

$$\frac{\mathcal{Y}}{E} = \left(\frac{F}{E}\right)^{1-\nu}\left[\sum_{z\in\mathbb{Z}}\sum_{\varphi\in\Phi}\frac{F_{z,\varphi}}{F}\left(\sum_{\widehat{n}\in\widehat{\mathbb{N}}}\sum_{h\in\mathbb{H}} \mathbf{Y}(\widehat{n}, h, z, \varphi) g_{z,\varphi}(\widehat{n}, h)\right)\right] \tag{A.19}$$

where

$$g_{z,\varphi}(\widehat{n}, h) \equiv \sum_{n_H=0}^{+\infty}\sum_{n_L=0}^{+\infty} \widetilde{f}_{z,\varphi}(n_H, n_L)\,\mathbf{1}\left[\left(\frac{n_H + n_L}{E/F} = \widehat{n}\right) \wedge \left(\frac{n_H}{n_H + n_L} = h\right)\right] \tag{A.20}$$

is the share of firms of type (z, φ) that have relative size $\widehat{n}$ and skill share h. This is equivalent to the result shown in equation (34).

[54] In the numerical implementation of the model in which there is an upper bound on employment, $\mathbb{H}$ is the union of finitely many finite sets, and therefore itself a finite set.

B First-Best Allocation of Workers to Firms

Consider a planner who can freely allocate workers to firms without being constrained by the search frictions present in the competitive equilibrium, but that takes as given the distribution of firms across productivity types (z, φ). In this section we show that, in such a "first-best" allocation of workers, the marginal product of each type of labor is equalized across firms. Hence, the allocation of high and low skilled workers is identical across firms of the same productivity type, (z, φ).

We explore two cases under which the planner allocates workers to firms of different productivities: (i) a case in which the measures of employed workers by skill type are fixed but total employment is not (Section B.1); and (ii) a case in which the total measure of employed workers is fixed but its split between skill types is not (Section B.2). This latter case is the one we use in the main text. Once again, in both cases we assume that the planner takes $F_{z,\varphi}$, the distribution of firms across productivity types (z, φ), as given.

B.1 Allocation of Workers when the Employment Measures by Skill are Fixed

For a given measure of active firms $F_{z,\varphi}$ and given measures of high skilled and low skilled workers E_H and E_H, we can obtain the first-best allocation of workers $n_H^*(z, \varphi)$ and $n_L^*(z, \varphi)$ across skill types (z, φ) as the solution to the following social planner problem:

$$\max_{\{n_H(z,\varphi), n_L(z,\varphi)\}} \sum_z \sum_\varphi Y\Big(n_H(z, \varphi), n_L(z, \varphi), z, \varphi\Big) F_{z,\varphi} \qquad \text{s.t.} \begin{cases} \sum_z \sum_\varphi n_H(z, \varphi) F_{z,\varphi} = E_H \\ \sum_z \sum_\varphi n_L(z, \varphi) F_{z,\varphi} = E_L \end{cases}$$

The FOC are given by $\partial Y(n_H, n_L, z, \varphi)/\partial n_H = \lambda_H$ and $\partial Y(n_H, n_L, z, \varphi)/\partial n_L = \lambda_L$, where $\lambda_H \geq 0$ and $\lambda_L \geq 0$ are the Lagrange multipliers associated to each constraint. Combining these two equations gives us the optimal ratio of each type of worker,

$$\frac{n_H(z, \varphi; \lambda_H, \lambda_L)}{n_L(z, \varphi; \lambda_H, \lambda_L)} = \left(\frac{\omega(\varphi)}{1 - \omega(\varphi)} \frac{\lambda_L}{\lambda_H}\right)^{\frac{1}{1-\alpha}} \tag{B.1}$$

which varies by φ but not by z, and varies by the aggregate relative scarcity of each type of worker given by λ_L/λ_H. Hence, in the first best all firms of the same type φ obtain the same ratio of high and low skilled workers. Let $E_H(\varphi) \equiv \sum_z n_H^*(z, \varphi) F_{z,\varphi}$ and $E_L(\varphi) \equiv \sum_z n_L^*(z, \varphi) F_{z,\varphi}$ be the total measures of high- and low-skilled workers allocated to firms of type φ. Then, using equation (B.1), the skill ratio at firms of type φ is given by

$$\frac{E_H(\varphi)}{E_L(\varphi)} = \left(\frac{\omega(\varphi)}{1 - \omega(\varphi)} \frac{\lambda_L}{\lambda_H}\right)^{\frac{1}{1-\alpha}}$$

and aggregating over firms of type φ it must be that

$$\frac{E_H}{E_L} = \sum_{\varphi} \frac{E_H(\varphi)}{E_L(\varphi)} \frac{F_\varphi}{F} = \sum_{\varphi} \left(\frac{\omega(\varphi)}{1-\omega(\varphi)} \frac{\lambda_L}{\lambda_H} \right)^{\frac{1}{1-\alpha}} \frac{F_\varphi}{F}$$

where $F_\varphi \equiv \sum_z F_{z,\varphi}$. Therefore,

$$\left(\frac{\lambda_H}{\lambda_L} \right)^{\frac{1}{1-\alpha}} = \frac{E_L}{E_H} \sum_{\varphi} \left(\frac{\omega(\varphi)}{1-\omega(\varphi)} \right)^{\frac{1}{1-\alpha}} \frac{F_\varphi}{F}$$

Hence, using equation (B.1):

$$\frac{n_H^*(z,\varphi)}{n_L^*(z,\varphi)} = \frac{\left(\frac{\omega(\varphi)}{1-\omega(\varphi)} \right)^{\frac{1}{1-\alpha}}}{\sum_{\varphi'} \left(\frac{\omega(\varphi')}{1-\omega(\varphi')} \right)^{\frac{1}{1-\alpha}} \frac{F_{\varphi'}}{F}} \left(\frac{E_H}{E_L} \right)$$

Therefore, the optimal skill share is a function of a firm's permanent type, but not of its idiosyncratic productivity: $h^*(\varphi) \equiv \frac{n_H^*(z,\varphi)}{n_H^*(z,\varphi)+n_L^*(z,\varphi)} = \left[1 + \frac{n_L^*(z,\varphi)}{n_H^*(z,\varphi)} \right]^{-1}$. Next, using the FOC of say n_H we obtain the employment demand:

$$n(z,\varphi;\lambda_H) = \left[\frac{\nu}{\lambda_H} \omega(\varphi) \Big(h^*(\varphi)\Big)^{\alpha-1} \left(\omega(\varphi) \Big(h^*(\varphi)\Big)^{\alpha} + (1-\omega(\varphi)) \Big(1-h^*(\varphi)\Big)^{\alpha} \right)^{\frac{\nu}{\alpha}-1} e^{z+\zeta(\varphi)} \right]^{\frac{1}{1-\nu}} \tag{B.2}$$

To characterize $n^*(z,\varphi)$ we need to get rid of λ_H. To do so, we aggregate equation (B.2):

$$E_H + E_L = \left(\frac{\nu}{\lambda_H} \right)^{\frac{1}{1-\nu}} \sum_z \sum_\varphi \left[\omega(\varphi) \Big(h^*(\varphi)\Big)^{\alpha-1} \left(\omega(\varphi) \Big(h^*(\varphi)\Big)^{\alpha} + (1-\omega(\varphi)) \Big(1-h^*(\varphi)\Big)^{\alpha} \right)^{\frac{\nu}{\alpha}-1} e^{z+\zeta(\varphi)} \right]^{\frac{1}{1-\nu}} F_{z,\varphi}$$

which gives us

$$n^*(z,\varphi) = \frac{(E_H+E_L) \left[\omega(\varphi) \Big(h^*(\varphi)\Big)^{\alpha-1} \left(\omega(\varphi) \Big(h^*(\varphi)\Big)^{\alpha} + (1-\omega(\varphi)) \Big(1-h^*(\varphi)\Big)^{\alpha} \right)^{\frac{\nu}{\alpha}-1} e^{z+\zeta(\varphi)} \right]^{\frac{1}{1-\nu}}}{\sum_{z'} \sum_{\varphi'} \left[\omega(\varphi') \Big(h^*(\varphi')\Big)^{\alpha-1} \left(\omega(\varphi') \Big(h^*(\varphi')\Big)^{\alpha} + (1-\omega(\varphi')) \Big(1-h^*(\varphi')\Big)^{\alpha} \right)^{\frac{\nu}{\alpha}-1} e^{z'+\zeta(\varphi')} \right]^{\frac{1}{1-\nu}} F_{z',\varphi'}}$$

the total employment of firm (z,φ) that is chosen by the planner.

B.2 Allocation of Workers when the Employment Measures by Skill are Not Fixed

We can also characterize the first-best allocation when the measure of active firms $F_{z,\varphi}$ and total employment E are given, but the split of E into E_H and E_L is kept free. This would imply solving the social planner problem:

$$\max_{\{n_H(z,\varphi),n_L(z,\varphi)\}} \sum_z \sum_\varphi Y(n_H,n_L,z,\varphi) F_{z,\varphi} \qquad \text{s.t.} \quad \sum_z \sum_\varphi n_H(z,\varphi) F_{z,\varphi} + \sum_z \sum_\varphi n_L(z,\varphi) F_{z,\varphi} = E_H + E_L$$

Compared to our case above, we now only have one constraint, and therefore a single Lagrange multiplier, λ. The ratio of FOCs already gives the optimal ratio of skilled to unskilled workers:

$$\frac{n_H^{**}(z,\varphi)}{n_L^{**}(z,\varphi)} = \left(\frac{\omega(\varphi)}{1-\omega(\varphi)}\right)^{\frac{1}{1-\alpha}} \tag{B.3}$$

which again implies that the optimal skill share is only a function of permanent productivity, $h^*(\varphi) = \Big(1 + n_L^{**}(z,\varphi)/n_H^{**}(z,\varphi)\Big)^{-1}$. Then, following the same derivations as above, we obtain the total employment of a firm of type (z,φ):

$$n^*(z,\varphi) = \frac{(E_H+E_L)\left[\omega(\varphi)\Big(h^*(\varphi)\Big)^{\alpha-1}\left(\omega(\varphi)\Big(h^*(\varphi)\Big)^{\alpha} + (1-\omega(\varphi))\Big(1-h^*(\varphi)\Big)^{\alpha}\right)^{\frac{\nu}{\alpha}-1} e^{z+\zeta(\varphi)}\right]^{\frac{1}{1-\nu}}}{\sum_{z'}\sum_{\varphi'}\left[\omega(\varphi')\Big(h^*(\varphi')\Big)^{\alpha-1}\left(\omega(\varphi')\Big(h^*(\varphi')\Big)^{\alpha} + (1-\omega(\varphi'))\Big(1-h^*(\varphi')\Big)^{\alpha}\right)^{\frac{\nu}{\alpha}-1} e^{z'+\zeta(\varphi')}\right]^{\frac{1}{1-\nu}} F_{z',\varphi'}} \tag{B.4}$$

C Numerical Appendix

C.1 Stationary Solution Algorithm

The idea of the algorithm is to loop over the fixed firm entry cost κ in order to find the fixed point that obtains the targeted measure of active firms, denoted F^*, as an outcome of the equilibrium. We solve the model on a grid $\overline{\mathcal{N}}_{OEH} \times \overline{\mathcal{N}}_{OEL} \times \overline{\mathcal{N}}_{FT} \times \mathbb{Z} \times \Phi$, where $\overline{\mathcal{N}}_{ij} \equiv \{0,1,2,\dots,N_{ij}\}$, $(i,j) \in \mathcal{I} \times \mathcal{J}$, for sufficiently large positive integer N_{ij}. The solution algorithm is as follows:

Step 0. Set $\iota = 0$. Choose guesses $\underline{\kappa}^{(0)} \in \mathbb{R}_+$ and $\overline{\kappa}^{(0)} \gg \underline{\kappa}^{(0)}$.

Step 1. At iteration $\iota \in \{0,1,2,\dots\}$, set the entry cost to:

$$\kappa^{(\iota)} = \frac{\underline{\kappa}^{(\iota)} + \overline{\kappa}^{(\iota)}}{2}$$

Step 2. Use Value Function Iteration to solve for $\boldsymbol{\Sigma}_{\varphi}^{(\iota)} \in \overline{\mathcal{N}}_{OEH} \times \overline{\mathcal{N}}_{OEL} \times \overline{\mathcal{N}}_{FT} \times \mathbb{Z}$:[55]

$$\begin{aligned}\boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n},z) = \frac{1}{\rho^{(\iota)}(\vec{n},z,\varphi)}\Bigg\{ & Y(\vec{n},z,\varphi) - \xi n_{FT}\big[p^{(\iota)}(\vec{n},z,\varphi)\big]^{\vartheta} + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\Bigg[n_{ij}\Big(\delta_{ij}^{(\iota)}(\vec{n},z,\varphi) + s_i^W + s^F\Big)\boldsymbol{U}^{(\iota)} \\ & - \chi n_{ij}\big[\delta_{ij}^{(\iota)}(\vec{n},z,\varphi)\big]^{\psi} + n_{ij}\Big(\delta_{ij}^{(\iota)}(\vec{n},z,\varphi) + s_i^W\Big)\boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}_{ij}^{-},z)\Bigg] \\ & + \sum_{i\in\mathcal{I}}\Bigg[\eta_i^{(\iota)}(\vec{n},z,\varphi)\max\Big(\boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}_{iL}^{+},z) - \boldsymbol{U}^{(\iota)}, \gamma\Big(\boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}_{iL}^{+},z) - \boldsymbol{U}^{(\iota)}\Big) + (1-\gamma)\boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n},z)\Big)\Bigg]\end{aligned}$$

[55]To arrive at this expression, we have used results (19) and (26) into equation (A.5).

$$+ n_{FT} p^{(\iota)}(\vec{n}, z, \varphi) \boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}^{p}, z) + n_{OEL} \tau \, \boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}^{\tau}, z) + \sum_{z'} \lambda(z'|z) \boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}, z') \Bigg\}$$

where:

$$\delta_{ij}^{(\iota)}(\vec{n}, z, \varphi) = \left[\frac{1}{\psi \chi} \left(\boldsymbol{U}^{(\iota)} + \boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}_{ij}^{-}, z) - \boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}, z) \right) \right]^{\frac{1}{\psi - 1}}$$

$$p^{(\iota)}(\vec{n}, z, \varphi) = \left[\frac{1}{\vartheta \xi} \left(\boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}^{p}, z) - \boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}, z) \right) \right]^{\frac{1}{\vartheta - 1}}$$

$$\eta_{i}^{(\iota)}(\vec{n}, z, \varphi) = A_{i}^{\frac{1}{\gamma}} \left[\left(\frac{1 - \gamma}{\rho \boldsymbol{U}^{(\iota)} - b} \right) \left(\boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}_{iL}^{+}, z) - \boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}, z) - \boldsymbol{U}^{(\iota)} \right) \right]^{\frac{1-\gamma}{\gamma}}$$

$$\rho^{(\iota)}(\vec{n}, z, \varphi) = \rho + s^{F} + n_{FT} p^{(\iota)}(\vec{n}, z, \varphi) + n_{OEL} \tau + \sum_{i \in \mathcal{I}} \left[\eta_{i}^{(\iota)}(\vec{n}, z, \varphi) + \sum_{j \in \mathcal{J}} n_{ij} \left(\delta_{ij}^{(\iota)}(\vec{n}, z, \varphi) + s_{i}^{W} \right) \right]$$

and where $\boldsymbol{U}^{(\iota)}$ is the solution to the free-entry condition:

$$\kappa^{(\iota)} = \gamma \left(\frac{\rho \boldsymbol{U}^{(\iota)} - b}{1 - \gamma} \right)^{\frac{\gamma - 1}{\gamma}} \sum_{\varphi \in \Phi} \sum_{z^{e} \in \mathbb{Z}} \pi_{\varphi}(\varphi) \pi_{z}(z^{e}) \sum_{i \in \mathcal{I}} \left[A_{i} \left(\boldsymbol{\Sigma}_{\varphi}^{(\iota)}(\vec{n}_{iL}^{e}, z^{e}) - \boldsymbol{U}^{(\iota)} \right) \right]^{\frac{1}{\gamma}}$$

where $\vec{n}_{i}^{e} = (n_{i}^{e}, \vec{n}_{-(ij)}^{e}) = (1, \vec{0})$.

Step 3. Compute the invariant distribution of firms and the aggregate measure of active firms $F^{(\iota)}$, as described in Appendix A.3.

Step 4. Compute $\Psi^{(\iota)} \equiv F^{(\iota)}/F^{*} - 1$. Stop if $\Psi^{(\iota)} \in [-\varepsilon, \varepsilon]$ for some small tolerance $\varepsilon > 0$. Otherwise,

(a) if $\Psi^{(\iota)} < -\varepsilon$, set $\underline{\kappa}^{(k+1)} = \underline{\kappa}^{(\iota)}$ and $\overline{\kappa}^{(k+1)} = \varrho \cdot \kappa^{(\iota)} + (1 - \varrho) \cdot \overline{\kappa}^{(\iota)}$, or

(b) if $\Psi^{(\iota)} > \varepsilon$, set $\underline{\kappa}^{(k+1)} = \varrho \cdot \kappa^{(\iota)} + (1 - \varrho) \cdot \underline{\kappa}^{(\iota)}$ and $\overline{\kappa}^{(k+1)} = \overline{\kappa}^{(\iota)}$,

where $\varrho \in (0, 1]$ is a dampening parameter, and go back to Step 1 with $[\iota] \leftarrow [\iota + 1]$.

C.2 Numerical Implementation of the Productivity Process

In the model, idiosyncratic productivity z is governed by a continuous-time Markov chain (CTMC), with associated infinitesimal generator matrix:

$$\boldsymbol{\Lambda}_{z} = \begin{pmatrix} -\sum_{j \neq 1} \lambda_{1j} & \lambda_{12} & \dots & \lambda_{1k} \\ \lambda_{21} & -\sum_{j \neq 2} \lambda_{2j} & \dots & \lambda_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{k1} & \lambda_{k2} & \dots & -\sum_{j \neq k} \lambda_{kj} \end{pmatrix}$$

where $\lambda_{ij} > 0$ is short-hand for $\lambda(z_j|z_i)$, $z_i, z_j \in \mathbb{Z}$. For the numerical implementation, we recover these rates by assuming an Ornstein-Uhlenbeck (OU) process for z_t (in logs):

$$d\ln(z_t) = -\rho_z \ln(z_t)dt + \sigma_z dB_t$$

where B_t is a Wiener process, and $\rho_z, \sigma_z > 0$.[56] This is a continuous-time process defined on a continuous support. To simulate such a process, and draw a one-to-one mapping between the (ρ_z, σ_z) parameters and the $\{\lambda_{ij}\}$ rates, we use the following steps:

1. First, we approximate the process in discrete time. For a given time interval $[0, T] \subset \mathbb{R}_+$, we partition the space into M subintervals of equal length $dt > 0$, i.e. $\mathbb{T} \equiv \{0 = t_0 < t_1 < \cdots < t_M = T\}$ with $t_{m+1} - t_m = dt$ and $dt = T/M$. As the model is calibrated at the quarterly frequency, dt represents a quarter. Then, we approximate the OU process using the Euler-Maruyama method:

$$\ln(z_k) = (1 - \rho_z dt)\ln(z_{k-1}) + \sigma_z \sqrt{dt}\, \varepsilon_k^z, \qquad \varepsilon_k^z \sim \text{i.i.d. } \mathcal{N}(0,1) \tag{C.1}$$

   for each $k = 1, \ldots, M$. From Ruiz-García (2021), we know that the autocorrelation coefficient and the dispersion in firm-level TFP in Spain is 0.81 and 0.34 at an annual frequency. Therefore, for persistence, we set $\rho_z = 1 - (0.81)^{1/4} = 0.0513$ for our quarterly calibration. Moreover, we compute a quarterly figure for dispersion from the yearly dispersion parameter as $\sigma_z = 0.34\left(\sum_{n=1}^{4}(0.81)^{(n-1)/2}\right)^{-1/2} = 0.1833$.

2. To estimate the discrete-time AR(1) process (C.1), we use the Tauchen (1986) method. The outcome of this method is a transition probability matrix $\mathbf{\Pi}_z = (\pi_{ij})$, where π_{ij} denotes the probability of a z_i-to-z_j transition in the $\mathbb{T}$ space.

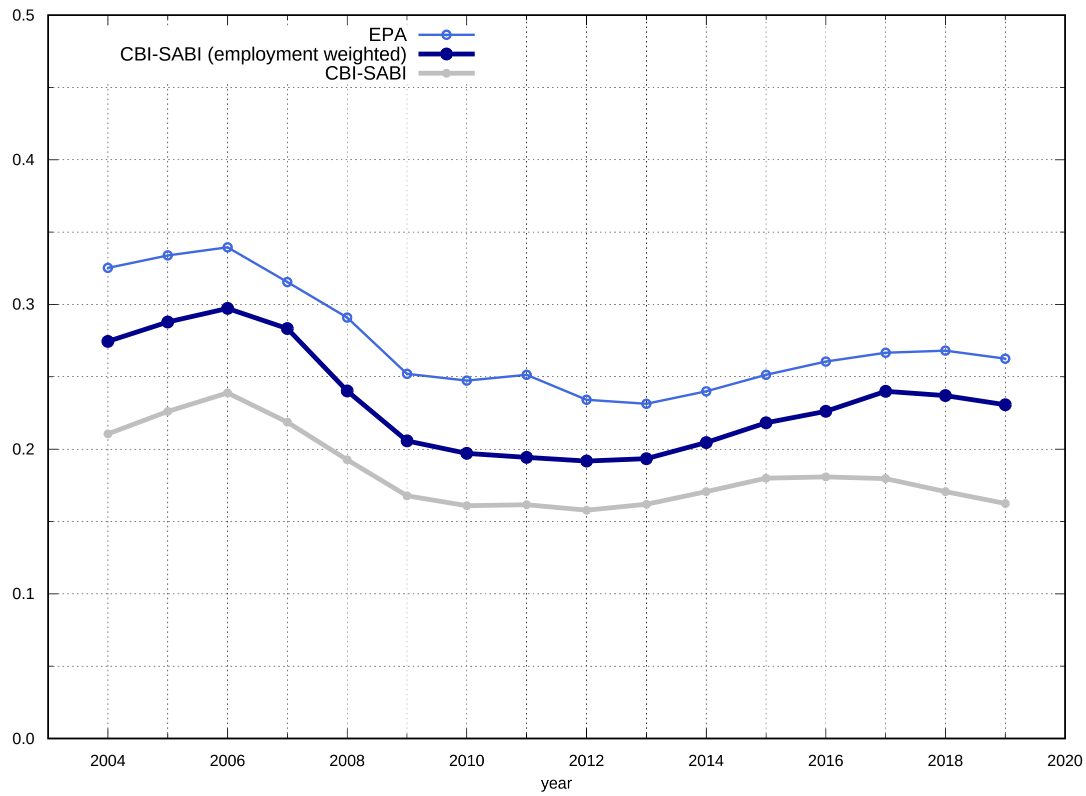
3. For the mapping from $\{\pi_{ij}\}$ transition probabilities to $\{\lambda_{ij}\}$ intensity rates, we use that any CTMC with generator matrix $\mathbf{\Lambda}_z$ maps into a discrete-time Markov chain with transition matrix $\mathbf{\Pi}_z(t)$ at time t in which holding times between arrivals are independently and exponentially distributed, so that $\mathbf{\Pi}_z(t) = e^{\mathbf{\Lambda}_z t}$. Then, we can solve for $\{\lambda_{ij}\}$ to obtain:

$$\lambda_{ij} = \begin{cases} -\frac{\pi_{ij}}{1-\pi_{ii}} \frac{\ln(\pi_{ii})}{dt} & \text{for } i \neq j \\ \frac{\ln(\pi_{ii})}{dt} & \text{otherwise} \end{cases}$$

[56] In levels, this is a diffusion of the type $dz_t = \mu(z_t)dt + \sigma(z_t)dB_t$, with $\mu(z) = z\left(-\rho_z \ln(z) + \frac{\sigma_z^2}{2}\right)$ and $\sigma(z) = \sigma_z z$.

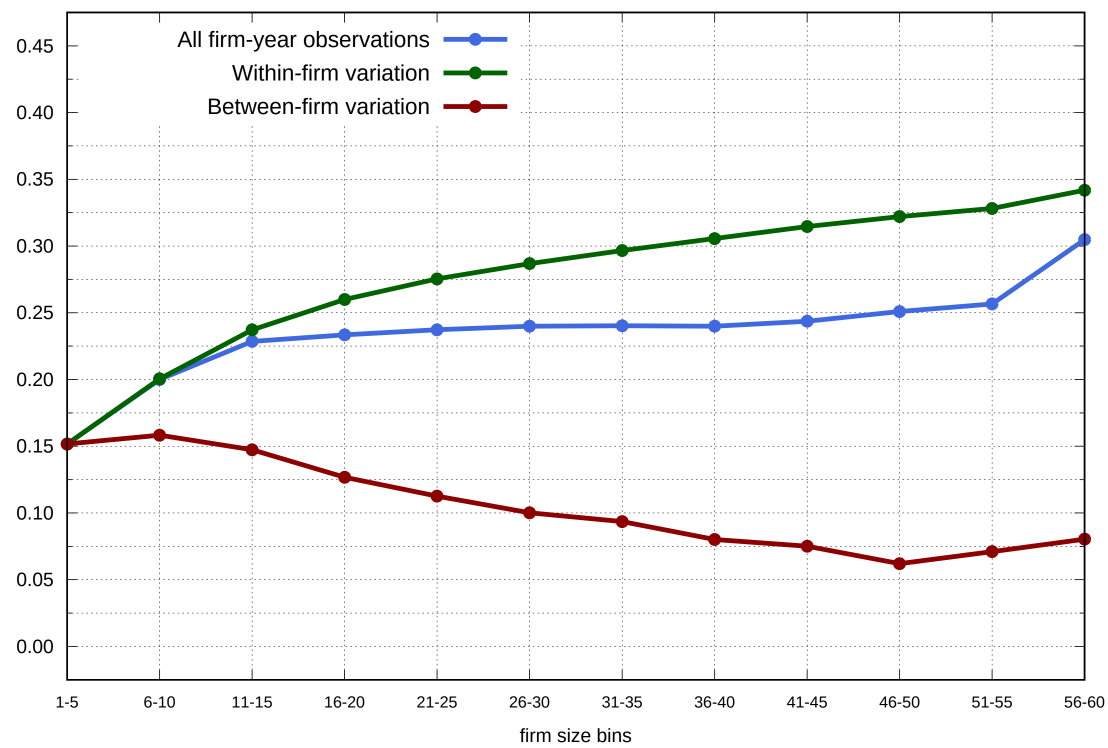
D Additional Figures

Figure D.1: Aggregate temporary share over time



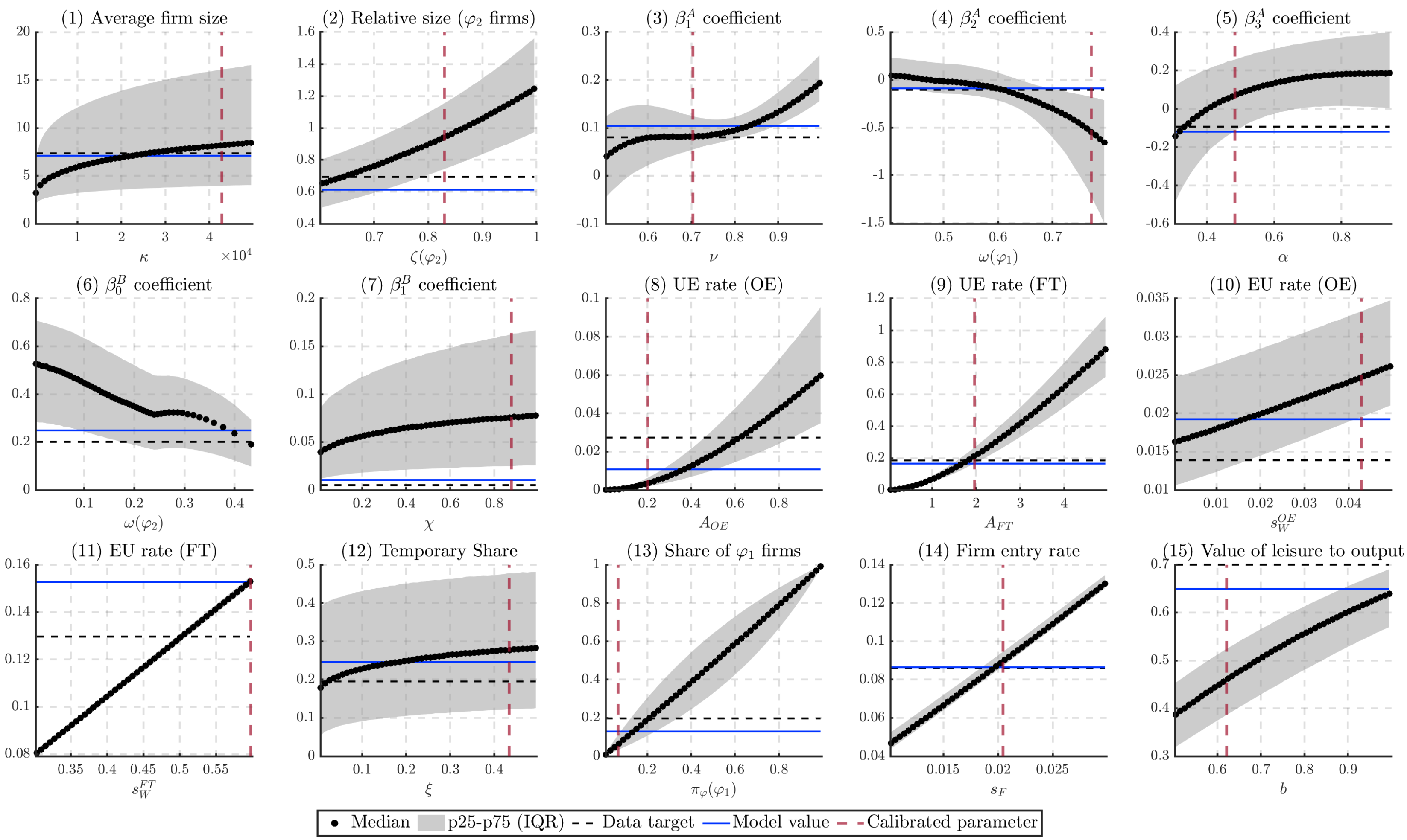
Notes: Light blue line: share of temporary workers in the Encuesta de Población Activa (the Spanish Labor Force Survey); grey line: average share of temporary workers across firms in our firm-level data; dark blue line: employment-weighted average share of temporary workers across firms in our firm-level data, which corresponds to the average share of temporary workers in the economy.

Figure D.2: Temporary share, by firm size, in the subsample.



Notes: This figure is the analogue of Figure 1 in the main text, but for the subsample used to calibrate the model.

Figure D.3: Global identification results.



Notes: Global identification results based on approximately 5 million model solutions. The black dots are the median of the distribution of each targeted moment for given value of the chosen parameter, generated from underlying random variation in all other parameters. The shaded areas are the 25th and 75th percentiles. The black dashed (respectively, blue solid) horizontal line is the data target (respectively, model prediction), and the red dashed vertical line is the calibrated value for the parameter.